USING THE METHOD OF MATCHED ASYMPTOTIC
EXPANSIONS, ANALYTICALLY INVESTIGATE THE
THREE-DIMENSIONAL, ATMOSPHERIC ENTRY
PROBLEM

THESIS

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DEPARTMENT OF THE AIR FORCE

AIR UNIVERSITY

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USING THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS, ANALYTICALLY INVESTIGATE THE THREE-DIMENSIONAL, ATMOSPHERIC ENTRY PROBLEM

THESIS

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Master of Science in Astronautical Engineering

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Preface

The purpose of this study was to determine if there existed an analytical solution for the three-body atmospheric entry equations. I decided to continue the excellent work by Drs. Vinh, Busemann, and Culp and use the Method of Matched Asymptotic Expansions to determine zero and first order solutions to the above equations. By using the computerized symbolic manipulator Mathematica™, the algebraic manipulations are reduced to a manageable level, so a more thorough analysis can be performed. By examining trends in these higher ordered solutions, I hoped a complete analytic solution, not evident from traditional analytic means, would become evident.

I would like to thank many people whose work aided my study. *First, I would like to recognize my faculty advisor, Capt Rodney Bain.* Without his keen mathematical insight , continuing motivation, and unfailing confidence, this study would not be possible. I am greatly indebted to the work of my predecessor, Harry Karasopoulos. Much of the groundwork of this study is based on Harry's master's thesis at AFIT. I would also like to thank Drs. Vinh, Busemann, and Culp, whose work in analytical flight mechanics greatly aided me in my study, and also Dr. Stephen Wolfram, whose Mathematica™ code saved me many late evenings from algebraic tedium. Finally I would like to thank my parents for their invariable motivation and support throughout my education. Without them, this study would not be possible.

<div align="right">Ted Masternak</div>

Table of Contents

List of Figures

Notation

Roman Letter Symbols

a	Acceleration (m/s^2)
B	Ballistic Coefficient
C_D	Drag Coefficient
C_L	Lift Coefficient
D	Drag (N)
g	Acceleration of Gravity (m/s^2)
g_s	Gravitational Acceleration at the Planet's Surface (m/s^2)
h	Non-Dimensional Altitude
I	Orbit Inclination Angle (deg or rad)
L	Lift (N)
L/D	Lift-to-Drag Ratio
m	Vehicle mass (kg)
q	Cosine of the Flight Path Angle, γ
r	Orbit Radius from Planet's Center (m)
r_s	Planetary Radius (m)
S	Aerodynamic Reference Area (m^2)
t	Time (sec)
u	Speed Ratio
V	Velocity (m/s)
y	Altitude (m)

Greek Letter Symbols

α	Argument of Latitude at Epoch (deg or rad)
β	Inverse Atmospheric Scale Height (m^{-1})
γ	Fiight Path Angle (deg or rad)
δ	Vehicle flight parameter
ϵ	Perturbation parameter
θ	Longitude (deg or rad)
λ	Vehicle flight parameter
μ	Planet Gravitational Parameter (m^3/s^2)
ξ	Magnified Non-Dimensional Altitude
ρ	Density (kg/m^3)
σ	Bank Angle (deg or rad)
ϕ	Latitude (deg or rad)
ψ	Heading Angle (deg or rad)
ω	Planet Rotation Rate (deg/s or rad/s)
Ω	Longitude of the Ascending Node (deg or rad)

Subscripts

0	Zero Order Expansion or Solution
1	First Order Expansion or Solution
2	Second Order Expansion
s	At the Surface of the Planet

Superscripts

i	Inner Expansion
c	Composite Expansion
o	Outer Expansion
$\hat{\ }$	Unit Direction Vector
$\dot{\ }$	Inertial Time Derivative

Abstract

Although numerical techniques exist to obtain solutions to highly non-linear and highly coupled systems, the trends and subtleties of the solution are frequently lost in the volume and form of tabular and graphical data in covering a wide range of initial conditions. By deriving an approximate, analytical solution, relationships between dependent parameters are discernable. Also, the derived solution is easily applied to any new set of initial conditions or can be modified to incorporate slightly different equations of motion. This study presents an analytical investigation of the three-dimensional equations of motion for lifting entry into a planetary atmosphere.

In this study, the equations of motion for lifting entry into a planetary atmosphere are derived. A non-rotating, spherical planet is assumed, as is a non-rotating, strictly exponential atmospheric model. The derived equations of motion are transformed to a variable set relating the classical orbital elements to the vehicle's altitude. Solutions to the resulting five non-linear, coupled, first order, ordinary differential equations are obtained by using the Method of Matched Asymptotic Expansions and a computerized symbolic manipulator, which performs the detailed algebraic computations. By using the planetary scale height-mean equatorial radius (PSHMER) product as a small parameter, both zero and first order expansions to the equations of motion are obtained.

It is demonstrated the analytical solution agrees with results obtained from numerical integration of the equations of motion. Due to approximations made in the solutions of the first order inner expansions, the analytical solution slightly deviates from the numerical solution at low vehicle altitudes. The two solutions are compared further and the validity of the analytical solution is examined.

USING THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS, ANALYTICALLY INVESTIGATE THE THREE-DIMENSIONAL, ATMOSPHERIC ENTRY PROBLEM

I.   <u>Introduction</u>

Since the recent explosion of readily available computing power, numerical solutions of complex systems described by non-linear phenomena have become ubiquitous. In using these numerical techniques, the trends and subtleties of the original system often become lost in the results of numerical analysis. By deriving a simple and relatively accurate analytical solution to a complex physical system, a readily available analysis becomes available which retains the subtleties of the original system

The equations of motion for lifting entry into a planetary atmosphere are highly coupled and non-linear. Instead of using a numerical technique to solve the equations of motion, an approximate analytical method can be used, where the trends and patterns of the original system are preserved and expressed in a solution which is simple, accurate and practical.

<u>Scope</u>

In this effort, the three-dimensional exact equations of motion for lifting entry into a non-rotating planet are developed to first order accuracy. Solutions to the equations of motion are developed by

1

considering the atmosphere as a boundary layer which perturbs the vehicle's motion as is approaches the planet's surface. The Method of Matched Asymptotic Expansions is used to develop zero and first order solutions valid throughout the flight trajectory of the entry body. Due to coordinate singularities, this study is limited to non-polar and non-equatorial entry trajectories.

Assumptions

In this study, the non-rotating planet is modeled as a spherical body possessing a inverse square law gravitational field. A strictly exponential, non-rotating atmosphere is assumed with its density as a function of the radial distance form the planet's surface. This investigation assumes the vehicle's trajectory is influenced only by aerodynamic forces and the planet's gravity. Other perturbing forces, such as solar/lunar gravitational forces and other celestial perturbations, are considered negligible. The vehicle's lift-to-drag ratio and ballistic coefficient are assumed constant and prescribed.

Methodology

In Section II, the equations of motion for three-dimensional, lifting entry for a spherical, non-rotating planet are derived from basic kinematic and force relationships. In Section III, these equations of motion undergo coordinate transformations to express the independent and dependent variables as convenient, non-dimensional orbital parameters. The first

order accurate analytical solutions are derived in Section IV using the Method of Matched Asymptotic Expansions. The five coupled, non-linear ordinary differential equations of motion are expanded to two separate, but overlapping domains: the outer, Keplerian, domain and the inner, atmospheric, domain. Each of these expansions is solved independently to create two sets of solutions to the original equations of motion. Finally the two solution sets are blended together to generate one solution valid over the entire vehicle trajectory. The validity of these analytical solutions is examined in Section V. It is shown the analytical solutions encounter singularities near polar orbits and slightly underestimate aerodynamic turning at low altitudes. Section VI summarizes the study's findings and presents recommendations for further study.

II. Derivation of the Equations of Motion

Introduction

In this section, the original equations of motion are derived for three-dimensional lifting entry into a non-rotating planetary atmosphere. The non-thrusting, lifting vehicle is modeled as a point mass in a three-dimensional coordinate space. The vehicle's orbit is assumed to be initially described by Keplerian or two-body motion. Aerodynamic forces are assumed to be the only perturbations acting on the entry vehicle.

For the sake of brevity, the derivations in this section are the abbreviated versions of much more detailed work presented in other studies (Vinh and others, 1980:20-28) and (Karasopoulos, 1988:9-36). Thorough descriptions of the derivations are presented in the two excellent references listed above.

Coordinate Systems

Figure 1 defines the planetary coordinate systems used in this study. In addition to coordinate systems referencing the vehicle to the planet, there is also present a coordinate system centered on the vehicle and relating its orientation to the planet below. Listed below are the coordinate system, their respective coordinate and unit vector representations and a brief description of each (Vinh and others, 1980:22-24).

4
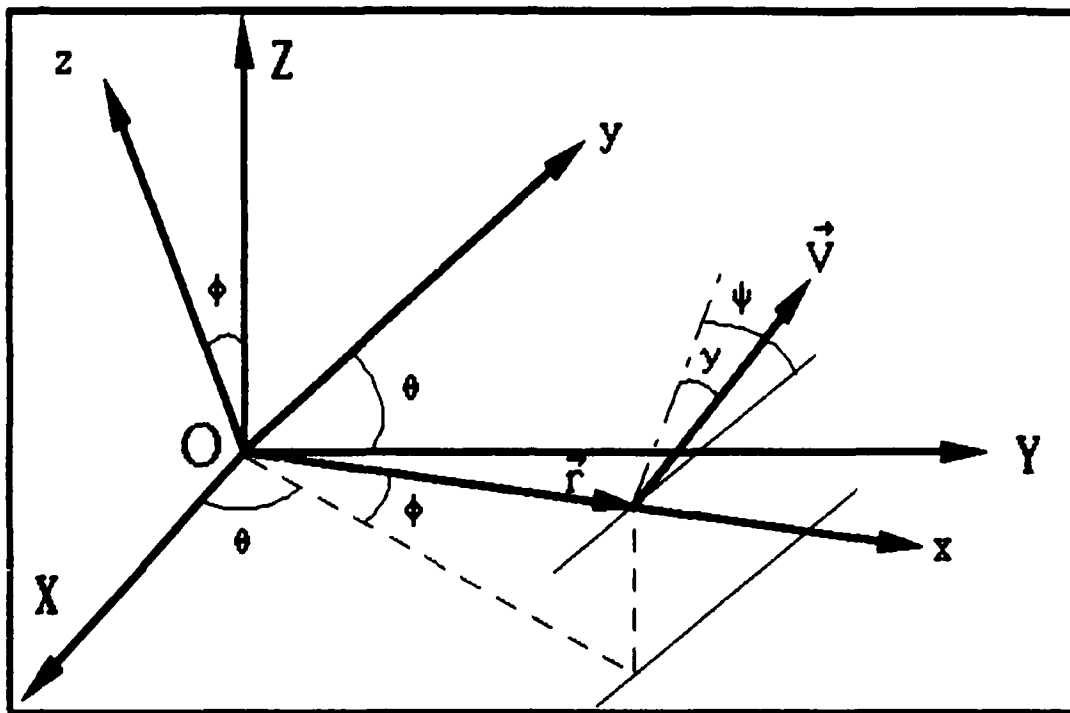
Figure 1.   Planet Centered, Rotating Coordinate System

Reference Coordinates	Unit Vectors	Rotation Rates	Description
$OX_IY_IZ_I$	$\hat{I}\,\hat{J}\,\hat{K}$	None	Inertial frame. Inertial reference frame whose center is coincident with the center of the spherical planet's gravitational field. The OX_IY_I plane is in the equatorial plane and the Z_I axis completes the right-handed system.
$OXYZ$	$\hat{i}_p\,\hat{j}_p\,\hat{k}_p$	$\omega\hat{k}_p$	Planet frame. Non-inertial reference frame whose center is coincident with the center of the spherical planet's gravitational field and fixed with respect to the planet. The OXY plane is in the equatorial plane and the Z axis completes the right-handed system. Thus, the coordinate system rotates at

5

the same rate as the planet, ω, about the $\hat{k}_P$ axis.

Oxyz	$\hat{i}_B \ \hat{j}_B \ \hat{k}_B$	$\dot{\theta}\hat{k}_P, \ \dot{\phi}\hat{j}_B$	Body Frame. Non-inertial reference frame whose center is located at the vehicle. The x-axis is along the position vector from point O to the vehicle, the y-axis is in the equatorial plane and orthogonal to the x-axis and the z-axis completes the right-handed system.
Ox'y'z'	$\hat{i}_W \ \hat{j}_W \ \hat{k}_W$	$\dot{\psi}\hat{i}_W, \ \dot{\gamma}\hat{k}_B$	Wind frame. Non-inertial reference frame whose center is located at the vehicle. The x-axis is along the lift vector (σ = 0) from point O to the vehicle, the y-axis is along the drag vector (σ = 0) and orthogonal to the x-axis and the z-axis completes the right-handed system.

Applying the coordinate transformations to the above systems results in two coordinate transformation matrices. Combined, the two transformations relate the vehicle's coordinate system in terms of the planet's rotating coordinate system.

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_P = \begin{bmatrix} \cos(\theta)\cos(\phi) & -\sin(\theta) & -\sin(\phi)\cos(\theta) \\ \sin(\theta)\cos(\phi) & \cos(\theta) & -\sin(\theta)\sin(\phi) \\ \sin(\phi) & 0 & \cos(\phi) \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_B \qquad (2.1)$$

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_B = \begin{bmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma)\cos(\psi) & \cos(\gamma)\cos(\psi) & -\sin(\psi) \\ -\sin(\gamma)\sin(\psi) & \cos(\gamma)\sin(\psi) & \cos(\psi) \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_W \qquad (2.2)$$

<u>Kinematic Equations</u>

6

The vehicle's position vector is given as $\vec{r} = r\,\hat{i}_B$. Differentiating this position vector to find the inertial time derivative of the position vector uses the vector differentiation relationship $\dfrac{^I d\vec{r}}{dt} = \dfrac{^B d\vec{r}}{dt} + \vec{\omega}^{BI} \times \vec{r}$, where

$$\vec{\omega}^{BI} = ((\omega + \dot{\theta})\sin(\phi))\,\hat{i}_B - \dot{\phi}\,\hat{j}_B + ((\omega + \dot{\theta})\cos(\phi))\,\hat{k}_B.$$

$$\frac{d\vec{r}}{dt} = \left(\frac{dr}{dt}\right)\hat{i}_B + \left(r\cos(\phi)\left(\frac{d\theta}{dt} + \omega\right)\right)\hat{j}_B + \left(r\,\frac{d\phi}{dt}\right)\hat{k}_B \qquad (2.3)$$

The vehicle's local velocity with respect to the Oxyz reference frame uses Eq (2.2) and is given as

$$\vec{V} = V\hat{j}_W = \left(V\sin(\gamma)\right)\hat{i}_B + \left(V\cos(\gamma)\cos(\psi)\right)\hat{j}_B + \left(V\cos(\gamma)\sin(\psi)\right)\hat{k}_B \qquad (2.4)$$

Equating Eqs (2.3) and (2.4) gives the kinematic equations as

$$\frac{dr}{dt} = V\sin(\gamma) \qquad (2.5)$$

$$\frac{d\theta}{dt} = \frac{V\cos(\gamma)\cos(\psi)}{r\cos(\phi)} - \omega \qquad (2.6)$$

$$\frac{d\phi}{dt} = \frac{V\cos(\gamma)\sin(\psi)}{r} \qquad (2.7)$$

Force Equations

Since the acceleration and the force on a vehicle are related, the acceleration of the vehicle is calculated in deriving the force equations.

Taking the inertial derivative of Eq (2.4), substituting in Eqs (2.6) and (2.7) and solving for the derivatives dV/dt, dγ/dt and dψ/dt gives

$$\frac{dV}{dt} = -\frac{F_T}{m} - g\sin(\gamma)$$

$$+\omega^2 r\cos(\phi)\left(\sin(\gamma)\cos(\phi) - \cos(\gamma)\sin(\phi)\sin(\psi)\right) \qquad (2.8)$$

$$V\frac{d\gamma}{dt} = \frac{F_N\cos(\sigma)}{m} - g\cos(\gamma) + \frac{V^2}{r}\cos(\gamma) + 2\omega V\cos(\phi)\cos(\psi)$$

$$+ \omega^2 r\cos(\phi)\left(\cos(\gamma)\cos(\phi) + \sin(\gamma)\sin(\phi)\sin(\psi)\right) \qquad (2.9)$$

$$V\frac{d\psi}{dt} = \frac{F_N\sin(\sigma)}{m\cos(\gamma)} - \frac{V^2}{r}\cos(\gamma)\cos(\psi)\tan(\phi) - \frac{\omega^2 r}{\cos(\gamma)}\sin(\phi)\cos(\phi)\sin(\psi)$$

$$+ 2\omega V\left(\tan(\gamma)\cos(\phi)\sin(\psi) - \sin(\phi)\right) \qquad (2.10)$$

F_N is defined as the lift force, F_T is defined as the aerodynamic and propulsive forces along the velocity vector and σ is called the bank angle, which rotates the F_N vector out of the local vertical plane.

Assumptions

The above three equations are known as the force equations. Since this study assumes a non-thrusting, lifting vehicle, the vehicle's thrust is zero or T - 0 and by the definitions of F_T and F_N, F_T - -D and F_N - L.

Since this study assumes the planet and its atmosphere are not rotating, ω- 0. This assumption is commonly used in entry flight mechanics, where the analysis is primarily concerned with the variations in

8

the velocity and altitude of the entry vehicle in the portion of the trajectory where high deceleration develops (Vinh and others, 1980:27). In his study, Karasopoulos showed exclusion of a rotating planet/atmosphere results in an analysis not valid for some entry trajectories, especially where the vehicle undergoes a long atmospheric entry period, as in very shallow entry trajectories. This study assumes the non-rotating equations of motion are valid and applies them to trajectories where they are accurate.

Substituting these assumptions in the above equations result in the modified kinematic and force equations.

$$\frac{dr}{dt} = V \sin(\gamma) \qquad (2.11)$$

$$\frac{d\theta}{dt} = \frac{V \cos(\gamma) \cos(\psi)}{r \cos(\phi)} \qquad (2.12)$$

$$\frac{d\phi}{dt} = \frac{V \cos(\gamma) \sin(\psi)}{r} \qquad (2.13)$$

$$\frac{dV}{dt} = -\frac{D}{m} - g \sin(\gamma) \qquad (2.14)$$

$$V \frac{d\gamma}{dt} = \frac{L \cos(\sigma)}{m} - g \cos(\gamma) + \frac{V^2}{r} \cos(\gamma) \qquad (2.15)$$

$$V \frac{d\psi}{dt} = \frac{L \sin(\sigma)}{m \cos(\gamma)} - \frac{V^2}{r} \cos(\gamma) \cos(\psi) \tan(\phi) \qquad (2.16)$$

Summary

This section briefly derives the equations of motion (the equations for flight over a non-rotating spherical planet) for a vehicle considered as a point mass traveling within a planetary atmosphere. The derived equations are repeated below for completeness.

$$\frac{dr}{dt} = V \sin(\gamma) \qquad (2.11)$$

$$\frac{d\theta}{dt} = \frac{V \cos(\gamma) \cos(\psi)}{r \cos(\phi)} \qquad (2.12)$$

$$\frac{d\phi}{dt} = \frac{V \cos(\gamma) \sin(\psi)}{r} \qquad (2.13)$$

$$\frac{dV}{dt} = -\frac{D}{m} - g \sin(\gamma) \qquad (2.14)$$

$$V \frac{d\gamma}{dt} = \frac{L \cos(\sigma)}{m} - g \cos(\gamma) + \frac{V^2}{r} \cos(\gamma) \qquad (2.15)$$

$$V \frac{d\psi}{dt} = \frac{L \sin(\sigma)}{m \cos(\gamma)} - \frac{V^2}{r} \cos(\gamma) \cos(\psi) \tan(\phi) \qquad (2.16)$$

10

III. Transformation of the Equations of Motion

Introduction

The equations of motion for three-dimensional, lifting entry into a non-rotating planetary atmosphere were derived in Section II. In this section, assumptions and approximations are defined. Coordinate transformations are undertaken to convert the equations of motion into a set of coupled ordinary differential equations with convenient dependent and independent variables   The equations of motion derived in Section II are given as

$$\frac{dr}{dt} = V \sin(\gamma) \tag{3.1}$$

$$\frac{d\theta}{dt} = \frac{V \cos(\gamma) \cos(\psi)}{r \cos(\phi)} \tag{3.2}$$

$$\frac{d\phi}{dt} = \frac{V \cos(\gamma) \sin(\psi)}{r} \tag{3.3}$$

$$\frac{dV}{dt} = -\frac{D}{m} - g \sin(\gamma) \tag{3.4}$$

$$V \frac{d\gamma}{dt} = \frac{L \cos(\sigma)}{m} - g \cos(\gamma) + \frac{V^2}{r} \cos(\gamma) \tag{3.5}$$

$$V \frac{d\psi}{dt} = \frac{L \sin(\sigma)}{m \cos(\gamma)} - \frac{V^2}{r} \cos(\gamma) \cos(\psi) \tan(\phi) \tag{3.6}$$

11

Assumptions

Spherical Planet/Atmosphere. Although planets are usually oblate ellipsoids, the spherical planet assumption is common in entry flight mechanics analysis, since the ellipticity of the planets are of the order 10^{-2} to 10^{-4} (Vinh and others, 1980:3). Thus, the error induced by this approximation is insignificant.

Following the spherical planet assumption, the planetary atmosphere is assumed to be a sphere. In reality, an atmosphere is approximately an oblate ellipsoid with minor deviations due to solar storms and uneven heating of the planetary atmosphere. Generally, these effects are present at altitudes above 250 kilometers, where aerodynamic forces are insignificant except in the slow orbital decay of satellites (Vinh and others, 1980:2).

Gravitational Model. Since the planet is being modeled as a sphere, its gravitational field follows Newton's inverse square law and is given as

$$g(h) = g_s \left(\frac{r_s}{r(h)}\right)^2 \qquad (3.7)$$

Atmospheric Density Model. The planet's atmosphere is assumed to be a non-rotating sphere fixed with respect to the planet. From the state equation for a gas and the hydrostatic equation, the planetary atmosphere is modeled by the equation

$$\frac{d\rho}{\rho} = -\beta \, dr \qquad (3.8)$$

12

where β is the local atmospheric density and r is the radial distance from the center of the planet. The $1/\beta$ term is called the scale height and describes the size of an altitude region where the atmosphere is accurately modeled by an exponential relation.

As in many other analytical entry flight mechanics studies, it is assumed the quantity βr is a constant for a planetary atmosphere (Vinh and others, 1980:5). For most planets, this value is usually of order 1000. For earth, it has an average value of about 900. Thus, solving Eq (3.8) with this assumption yields

$$\rho = \rho_s \left(\frac{r}{r_s}\right)^{-\beta r} \tag{3.9}$$

This study assumes the planetary atmosphere is accurately modeled by a exponential atmospheric model given as

$$\rho = \rho_s e^{-\beta y} \tag{3.10}$$

where y is the altitude above the planet surface. By definition, $h = y/r_s$ Thus $\beta y = \beta h r_s = h/\epsilon$ or

$$\rho = \rho_s e^{-h/\epsilon} \tag{3.11}$$

where

$$\epsilon = \frac{1}{\beta r_s} \tag{3.12}$$

The value of ϵ is on the order of $1/1000$ and thus is a small number. Using this variable as a small parameter allows the use of Matched Asymptotic

13

Expansions to be applied to derive an analytical solution to the equations of motion.

Aerodynamic Forces. The lift and drag parameters for the entry vehicle are given by the following relationships, which incorporate the exponential atmospheric density relation determined above.

$$L = C_L \frac{\rho V^2 S}{2} = C_L \frac{\rho_s e^{-h/\epsilon} V^2 S}{2} \tag{3.13}$$

$$D = C_D \frac{\rho V^2 S}{2} = C_D \frac{\rho_s e^{-h/\epsilon} V^2 S}{2} \tag{3.14}$$

Although C_L and C_D are functions of the angle of attack, Mach number and other flight parameters, this study assumes they are constant and prescribed at the beginning of the entry trajectory. This is a common assumption when dealing with hypersonic flight mechanics (Vinh and others, 1980:101).

Substitution of these assumptions into the equations of motion given by Eqs (3.1)-(3.6) gives

$$\frac{dr}{dt} = V \sin(\gamma) \tag{3.15}$$

$$\frac{d\theta}{dt} = \frac{V \cos(\gamma) \cos(\psi)}{r \cos(\phi)} \tag{3.16}$$

$$\frac{d\phi}{dt} = \frac{V \cos(\gamma) \sin(\psi)}{r} \tag{3.17}$$

$$\frac{dV}{dt} = - \frac{C_D \rho_s e^{-h/\epsilon} V^2 S}{2m} - g_s \left(\frac{r_s}{r}\right)^2 \sin(\gamma) \tag{3.18}$$

14

$$V \frac{d\gamma}{dt} = \frac{C_L \rho_s e^{-h/\epsilon} V^2 S \cos(\sigma)}{2m} - g_s \left(\frac{r_s}{r}\right)^2 \cos(\gamma) + \frac{V^2}{r} \cos(\gamma) \qquad (3.19)$$

$$V \frac{d\psi}{dt} = \frac{C_L \rho_s e^{-h/\epsilon} V^2 S \sin(\sigma)}{2m \cos(\gamma)} - \frac{V^2}{r} \cos(\gamma) \cos(\psi) \tan(\phi) \qquad (3.20)$$

Transformation of the Independent Variable

As given in Eqs (3.1)-(3.5), the equations of motion are in terms of the dimensional variable time, t. For atmospheric entry, it is more convenient to relate the motion of the vehicle in terms of the orbital altitude, r, since the two major forces controlling the entry trajectory, gravity and lift/drag, are functions of r. To change the integration variable from t to r, the chain rule for differentiation is used. Thus, using dr/dt relationship from Eq (3.1), differentiation with respect to r is defined as

$$\frac{d}{dr} = \frac{d}{dt} \frac{dt}{dr} = \frac{1}{V \sin(\gamma)} \frac{d}{dt} \qquad (3.21)$$

This transformation reduces the number of equations of motion from six to five since Eq (3.1) is incorporated into the other five equations by the above use of the chain rule.

In atmospheric entry analysis, a dimensionless altitude, h, is commonly used as the independent variable of integration for the equations of motion. 'h' is defined as (Vinh and others, 1980:256)

$$h = \frac{y}{r_s} \qquad (3.22)$$

15

Since r is currently the integration variable for the equations of motion, a transformation relationship is needed to convert r to h. By definition

$$r = r_s + y = r_s (1 + h) \tag{3.23}$$

Differentiation the above equation gives

$$\frac{dr}{dh} = r_s$$

Again using the chain rule for differentiation yields

$$\frac{d}{dh} = \frac{d}{dr}\frac{dr}{dh} = r_s\frac{d}{dr} \tag{3.24}$$

Applying the above definition for r as well as the differentiation transformation for d/dr yields

$$\frac{d}{dh} = r_s\frac{d}{dr} = \frac{r_s}{V\sin(\gamma)}\frac{d}{dt} \tag{3.25}$$

Applying this transformation, and the definition of r, to Eqs (3.15)-(3.20) gives the modified equations of motion as

$$\frac{d\theta}{dh} = \frac{\cos(\psi)}{(1+h)\tan(\gamma)\cos(\phi)} \tag{3.26}$$

$$\frac{d\phi}{dh} = \frac{\sin(\psi)}{(1+h)\tan(\gamma)} \tag{3.27}$$

$$\frac{dV}{dh} = -\frac{C_D\, r_s\, \rho_s e^{-h/\epsilon}\, VS}{2m\sin(\gamma)} - \frac{g_s\, r_s}{V(1+h)^2} \tag{3.28}$$

16

$$\frac{d\gamma}{dh} = \frac{C_L\, r_s\, \rho_s e^{-h/\epsilon}\, S \cos(\sigma)}{2m \sin(\gamma)} - \frac{g_s\, r_s}{(1 + h)^2\, V^2 \tan(\gamma)} + \frac{1}{(1 + h) \tan(\gamma)} \quad (3.29)$$

$$\frac{d\psi}{dh} = \frac{C_L\, r_s\, \rho_s e^{-h/\epsilon}\, S \sin(\sigma)}{2m \sin(\gamma) \cos(\gamma)} - \frac{\cos(\psi)\, \tan(\phi)}{(1 + h) \tan(\gamma)} \quad (3.30)$$

Thus, the above modified equations of motion are expressed in a convenient integration parameter, h. In the next section, the dependent variables are transformed into a set of orbital parameters which are convenient to use in the analytical analysis performed in Section IV.

Transformation to Orbital Elements

Ballistic Coefficient and Other Flight Parameters. In Eqs (3.26)-(3.30), there are several lengthy constant coefficients which pertain to the flight vehicle and the planetary atmosphere it is entering. By developing non-dimensional shorthand expressions for some of these terms, the equations become easier to manipulate. The first non-dimensional term defined is called the ballistic coefficient, B, and specifies physical characteristics of the flight vehicle, as well as the planetary atmosphere it is entering. B is assumed constant throughout the entry trajectory, specified by initial conditions and defined as (Vinh and others, 1980:256)

$$B = \frac{\rho_s S C_D}{2m\beta} = \frac{\rho_s r_s S C_D \epsilon}{2m} \quad (3.31)$$

To help further facilitate the manipulation of the equations of motion the terms specifying the bank angle, σ, and the lift-to-drag ratio, C_L/C_D, are

17

combined. δ and λ are assumed constant throughout the entry trajectory, specified by initial conditions and defined as (Vinh and others, 1980:255)

$$\lambda = \frac{C_L}{C_D} \cos(\sigma) \qquad (3.32)$$

$$\delta = \frac{C_L}{C_D} \sin(\sigma) \qquad (3.33)$$

Substituting the above three relations into Eqs (3.26)-(3.30) gives

$$\frac{d\theta}{dh} = \frac{\cos(\psi)}{(1 + h) \tan(\gamma) \cos(\phi)} \qquad (3.34)$$

$$\frac{d\phi}{dh} = \frac{\sin(\psi)}{(1 + h) \tan(\gamma)} \qquad (3.35)$$

$$\frac{dV}{dh} = -\frac{BVe^{-h/\epsilon}}{\epsilon \sin(\gamma)} - \frac{g_s r_s}{V(1 + h)^2} \qquad (3.36)$$

$$\frac{d\gamma}{dh} = \frac{B\lambda e^{-h/\epsilon}}{\epsilon \sin(\gamma)} - \frac{g_s r_s}{(1 + h)^2 V^2 \tan(\gamma)} + \frac{1}{(1 + h) \tan(\gamma)} \qquad (3.37)$$

$$\frac{d\psi}{dh} = \frac{B\delta e^{-h/\epsilon}}{\epsilon \sin(\gamma) \cos(\gamma)} - \frac{\cos(\psi) \tan(\phi)}{(1 + h) \tan(\gamma)} \qquad (3.38)$$

Modified Speed Ratio. Recent analytical flight mechanics studies have determined that transforming the velocity terms, V and dV/dh, into the modified Chapman variable, u, places the equations of motion into a more practical form. By definition, u is defined as (Vinh and others, 1980:229)

$$u - \frac{V^2 \cos^2(\gamma)}{gr} - \frac{V^2(1+h)\cos^2(\gamma)}{g_s r_s} \tag{3.39}$$

Solving for V yields

$$V - \frac{1}{\cos(\gamma)} \sqrt{\frac{u g_s r_s}{(1+h)}} \tag{3.40}$$

As a convenience u is also known as the speed ratio since it relates the vehicle's local horizontal velocity to the circular orbital velocity at the vehicle's current altitude (Karasopoulos, 1988:45). To replace V with u in Eqs (3.34)-(3.38), Eq (3.39) is differentiated with respect to h, keeping g_s and r_s constant.

$$\frac{du}{dh} - \frac{V^2 \cos^2(\gamma)}{g_s r_s} + \frac{2V(1+h)\cos^2(\gamma)}{g_s r_s} \frac{dV}{dh}$$

$$- \frac{2V^2(1+h)\sin(\gamma)\cos(\gamma)}{g_s r_s} \frac{d\gamma}{dh} \tag{3.41}$$

Substituting Eqs (3.36) and (3.37) into the above equation transforms the dV/dh equation of motion into du/dh form and replaces Eq (3.36)

$$\frac{du}{dh} - - \frac{u}{(1+h)} - \frac{2Bu\left(1+\lambda \tan(\gamma)\right)}{\epsilon \sin(\gamma)} e^{-h/\epsilon} \tag{3.42}$$

Modified Flight Path Angle. In Eq (3.37), $\sin(\gamma)$ terms appear in the denominator. Most realistic entry trajectories begin with $-10° \le \gamma < 0°$ and often result in aerodynamic skipping, where γ switches from a negative to positive quantity. During this transition from positive to negative values, $\gamma \sim 0°$, which could create a singularity in Eq (3.37) since $1/\sin(\gamma)$, where $\gamma -$

19

0°, is not defined. To eliminate this potential singularity as well as to simplify the equations of motion, the flight path angle, γ, is transformed to the variable q, given as (Vinh and others, 1980:257)

$$q = \cos(\gamma) \tag{3.43}$$

Differentiating the above relation with respect to h yields

$$\frac{dq}{dh} = -\sin(\gamma)\frac{d\gamma}{dh} \tag{3.44}$$

Using the above equation and the definition for the speed ratio transforms Eq (3.37) into a differential equation for q.

$$\frac{dq}{dh} = -\frac{q}{1+h}\left(1 - \frac{q^2}{u}\right) - \frac{B\lambda\,e^{-h/\epsilon}}{\epsilon} \tag{3.45}$$

Thus, the equations of motion are now given as

$$\frac{du}{dh} = -\frac{u}{(1+h)} - \frac{2Bu\left(1+\lambda\,\tan(\gamma)\right)}{\epsilon\sin(\gamma)}e^{-h/\epsilon} \tag{3.46}$$

$$\frac{dq}{dh} = -\frac{q}{1+h}\left(1 - \frac{q^2}{u}\right) - \frac{B\lambda\,e^{-h/\epsilon}}{\epsilon} \tag{3.47}$$

$$\frac{d\theta}{dh} = \frac{\cos(\psi)}{(1+h)\,\tan(\gamma)\,\cos(\phi)} \tag{3.48}$$

$$\frac{d\phi}{dh} = \frac{\sin(\psi)}{(1+h)\,\tan(\gamma)} \tag{3.49}$$

$$\frac{d\psi}{dh} = \frac{B\delta\,e^{-h/\epsilon}}{\epsilon\sin(\gamma)\cos(\gamma)} - \frac{\cos(\psi)\,\tan(\phi)}{(1+h)\,\tan(\gamma)} \tag{3.50}$$

where $q = \cos(\gamma)$.

Classical Orbital Elements. Before using the Method of Matched
Asymptotic Expansions to analytically solve the above equations, they are
transformed again to express them in a form which will simplify the
analysis found in the next section. Currently, the independent variables are
the speed ratio (u), modified flight path angle (q), longitude (θ), latitude (ϕ)
and heading angle (ψ). During the orbital lifetime of a satellite, these
variables are constantly changing. By transforming the variable set (θ, ϕ
and ψ) to the set of classical orbital elements (I, Ω and α), the resulting
analysis is greatly simplified. This simplification results from the classical
orbital elements being constant for two-body motion, where there are no
perturbing forces (Wiesel, 1989:34-35, 58). As shown in the next section,
the equations of motion will be analyzed in two domains, exo-atmospheric
and atmospheric flight. This transformation will greatly simplify the exo-
atmospheric (two-body) analysis, since two of the equations of motion are
constant.

Inclination Angle, I. To derive the equation of motion
describing the inclination angle, the spherical trigonometric relationship
(A.7) is used and given below.

$$\cos(I) = \cos(\psi)\cos(\phi) \tag{3.51}$$

Differentiating the above equation with respect to h and solving for
dI/dh gives

$$\frac{dI}{dh} = \frac{\sin(\phi)\cos(\psi)}{\sin(I)}\frac{d\phi}{dh} + \frac{\cos(\phi)\sin(\psi)}{\sin(I)}\frac{d\psi}{dh} \tag{3.52}$$

21

Substituting in Eqs (3.49) and (3.50) for the differential relations dϕ/dh and dψ/dh and using the spherical trigonometric equations (A.5) and (A.11) gives

$$\frac{dI}{dh} = \frac{B\,\delta\,\cos(\alpha)}{\epsilon\,\sin(\gamma)\cos(\gamma)}\,e^{-h/\epsilon} \qquad (3.53)$$

Longitude of the Ascending Node, Ω. To derive the equation of motion describing the longitude of the ascending node, the spherical trigonometric relationship (A.15) is used and given below.

$$\sin(\psi) = \sin(I)\cos(\theta - \Omega) \qquad (3.54)$$

Differentiating the above equation with respect to h and solving for dΩ/dh gives

$$\frac{d\Omega}{dh} = \frac{\cos(\psi)}{\sin(\theta-\Omega)\sin(I)}\frac{d\psi}{dh} - \frac{\cos(\theta-\Omega)\cos(I)}{\sin(\theta-\Omega)\sin(I)}\frac{dI}{dh} + \frac{d\phi}{dh} \qquad (3.55)$$

Substituting in Eqs (3.49), (3.50) and (3.53) for the differential relations dϕ/dh, dψ/dh and dI/dh and using the spherical trigonometric equations (A.4), (A.10) and (A.11) gives

$$\frac{d\Omega}{dh} = \frac{B\,\delta\,\sin(\alpha)}{\epsilon\,\sin(I)\,\sin(\gamma)\,\cos(\gamma)}\,e^{-h/\epsilon} \qquad (3.56)$$

Argument of Latitude at Epoch, α. To derive the equation of motion describing the argument of latitude at epoch, the spherical trigonometric relationship (A.4) is used and given below.

$$\sin(\alpha) = \frac{\sin(\phi)}{\sin(I)} \qquad (3.57)$$

22

Differentiating the above equation with respect to h and solving for dα/dh gives

$$\frac{d\alpha}{dh} = \frac{\cos(\phi)}{\cos(\alpha)\sin(I)}\frac{d\phi}{dh} - \frac{\sin(\alpha)\cos(I)}{\cos(\alpha)\sin(I)}\frac{dI}{dh}$$
(3.58)

Substituting in Eqs (3.49) and (3.53) for the differential relations dφ/dh and dI/dh and using the spherical trigonometric equation (A.11) gives

$$\frac{d\alpha}{dh} = \frac{1}{(1+h)\tan(\gamma)} - \frac{B\,\delta\,\sin(\alpha)}{\epsilon\,\tan(I)\,\sin(\gamma)\,\cos(\gamma)}e^{-h/\epsilon}$$
(3.59)

Summary

The above derivations develop the three-dimensional equations of motion for atmospheric entry. They relate the classical orbital elements to the non-dimensional altitude.

$$\frac{du}{dh} = -\frac{u}{(1+h)} - \frac{2Bu\left(1+\lambda\tan(\gamma)\right)}{\epsilon\sin(\gamma)}e^{-h/\epsilon}$$
(3.60)

$$\frac{dq}{dh} = \frac{q}{(1+h)}\left(\frac{q^2}{u}-1\right) - \frac{B\,\lambda\,e^{-h/\epsilon}}{\epsilon}$$
(3.61)

$$\frac{dI}{dh} = \frac{B\,\delta\,\cos(\alpha)}{\epsilon\,\sin(\gamma)\cos(\gamma)}e^{-h/\epsilon}$$
(3.62)

$$\frac{d\Omega}{dh} = \frac{B\,\delta\,\sin(\alpha)}{\epsilon\,\sin(I)\,\sin(\gamma)\,\cos(\gamma)}e^{-h/\epsilon}$$
(3.63)

$$\frac{d\alpha}{dh} = \frac{1}{(1+h)\tan(\gamma)} - \frac{B\,\delta\,\sin(\alpha)}{\epsilon\,\tan(I)\,\sin(\gamma)\,\cos(\gamma)}\,e^{-h/\epsilon}$$

(3.64)

IV. Solutions to the Equations of Motion Using Matched Asymptotic Expansions

In Section III, the five equations of motion for three-dimensional atmospheric entry were derived and are repeated below.

$$\frac{du}{dh} = -\frac{u}{(1+h)} - \frac{2Bu\left(1+\lambda\,\tan(\gamma)\right)}{\epsilon\,\sin(\gamma)}\,e^{-h/\epsilon} \tag{4.1}$$

$$\frac{dq}{dh} = \frac{q}{(1+h)}\left(\frac{q^2}{u} - 1\right) - \frac{B\,\lambda\,e^{-h/\epsilon}}{\epsilon} \tag{4.2}$$

$$\frac{dI}{dh} = \frac{B\,\delta\,\cos(\alpha)}{\epsilon\,\sin(\gamma)\cos(\gamma)}\,e^{-h/\epsilon} \tag{4.3}$$

$$\frac{d\Omega}{dh} = \frac{B\,\delta\,\sin(\alpha)}{\epsilon\,\sin(I)\,\sin(\gamma)\,\cos(\gamma)}\,e^{-h/\epsilon} \tag{4.4}$$

$$\frac{d\alpha}{dh} = \frac{1}{(1+h)\,\tan(\gamma)} - \frac{B\,\delta\,\sin(\alpha)}{\epsilon\,\tan(I)\,\sin(\gamma)\,\cos(\gamma)}\,e^{-h/\epsilon} \tag{4.5}$$

The above five differential equations are first order, non-linear ordinary differential equations (ODEs). Although there exist techniques to solve any first order, linear ODE, the non-linearity and coupling among the five equations prevents them from being solved by traditional analytical means (Rainville and Bedient, 1981:36). To preclude a numerical solution to the above ODEs, and thus retaining some insight into the mechanics of the problem, a higher order analytical solution is used (Vinh and others,

25

1980:254). The solution method implemented is called the Method of Matched Asymptotic Expansions.

The Method of Matched Asymptotic Expansions

The problem of atmospheric entry is fundamentally a situation where an initially small perturbation, the atmosphere, is introduced into the equations of motion, but its exponentially increasing effect causes the orbital elements to undergo very rapid changes over a very narrow region of the independent variable, the vehicles's altitude (Nayfeh, 1981:270-279). Thus, the atmospheric entry equations are modeled as if the atmosphere acts as a boundary layer adjacent to the orbital region solely specified by Keplerian, two-body, motion, neglecting other perturbing forces (Vinh and others, 1980:259). From the perspective of the entry vehicle, its orbit is initially prescribed by gravitational forces, but as its altitude decreases, it enters a boundary layer region formed by the atmosphere. The aerodynamic forces of lift and drag will vary drastically over a small range of altitude, as compared to the mean orbit altitude, and thus dramatically alter the orbital parameters.

The Method of Matched Asymptotic Expansions is chosen over other analytical techniques such as the methods of multiple scales and straight-forward expansions since the Method of Matched Asymptotic Expansions is more adept at handling non-linear differential equations (Nayfeh, 1981:279). In this method, the solution to the problem is represented by two expansions, each of which is valid in part of the problem domain, either exo-atmospheric or atmospheric altitude. Since the two expansions have

some overlap, they are matched to create one composite expansion valid over the entire problem domain.

Deriving the Outer and Inner Expansions Using Mathematica™

The process of deriving the outer and inner expansions using Matched Asymptotic Expansions is a very laborious and tedious exercise in algebraic bookkeeping. To expedite these derivations, a computerized, symbolic manipulator, Mathematica™, is used to perform the outer and inner expansions. Using Mathematica™ decreases the time performing algebraic manipulations. Thus, more time is spent in analysis of the derived solutions. The computer code .-ε J in this study is presented in Appendix B, as is the methodology used ιo develop the code.

In the following expansions, two of the ten expansions are performed to illustrate the techniques involved. The remaining expansions are presented without derivation and are as given from Mathematica™ output.

Outer Expansions

The solutions developed for the exo-atmospheric (non-boundary layer) portion of the domain are called the outer solutions. These solutions are developed from asymptotic expansions of the equations of motion using the small parameter, ϵ. The outer solution variables are denoted by the superscript "o" and are assumed as follows (Vinh and others, 1980:259):

$$u^o = u_0(h) + u_1(h)\epsilon + u_2(h)\epsilon^2 + O(\epsilon^3)$$

27

$$q^0 = q_0(h) + q_1(h)\epsilon + q_2(h)\epsilon^2 + O(\epsilon^3)$$

$$I^0 = I_0(h) + I_1(h)\epsilon + I_2(h)\epsilon^2 + O(\epsilon^3)$$

$$\Omega^0 = \Omega_0(h) + \Omega_1(h)\epsilon + \Omega_2(h)\epsilon^2 + O(\epsilon^3)$$

$$\alpha^0 = \alpha_0(h) + \alpha_1(h)\epsilon + \alpha_2(h)\epsilon^2 + O(\epsilon^3)$$

$$\gamma^0 = \gamma_0(h) + \gamma_1(h)\epsilon + \gamma_2(h)\epsilon^2 + O(\epsilon^3) \qquad (4.6)$$

The approximate solutions for lifting atmospheric entry are of order ϵ^0 (Karasopoulos, 1988:82). The solutions derived to ϵ^1 and higher orders of ϵ act as correcting factors to the zero order solutions and account for decreasingly significant physical characteristics of the problem. This is evident in Eq (4.6), since the solution order (u_1, u_2,..., for example) is multiplied by corresponding powers of ϵ. Thus, the zero order solution gives the primary behavior of the variable and higher order solutions add corrections which progressively bring this approximate solution in line with the actual solution.

du/dh Outer Expansion. This expansion is performed to illustrate the techniques used in deriving an expansion. The original ODE is given in Eq (4.1) as

$$\frac{du}{dh} = -\frac{u}{(1+h)} - \frac{2Bu\left(1+\lambda\,\tan(\gamma)\right)}{\epsilon\,\sin(\gamma)}e^{-h/\epsilon}$$

Next, straightforward expansions for the variables in the above ODE are defined. The expansions for u and γ are assumed in Eq (4.6) while templates for $\tan(\gamma)$ and $1/\sin(\gamma)$ are derived in Appendix C and given as

28

Eqs (C.11) and (C.13). Substituting the expansion for y in the two templates gives

$$\left[\sin\left(y_0 + y_1\epsilon + y_2\epsilon^2 + O(\epsilon^3)\right)\right]^{-1}$$

$$= \frac{1}{\sin(y_0)} - \frac{y_1\cos(y_0)}{\sin^2(y_0)}\epsilon + \left(\frac{y_1^2\left(1 + \cos^2(y_0)\right)}{2\sin^3(y_0)} - \frac{y_2\cos(y_0)}{\sin^2(y_0)}\right)\epsilon^2 + O(\epsilon^3)$$

$$\tan(y_0 + y_1\epsilon + y_2\epsilon^2 + O(\epsilon^3)) = \tan(y_0) + \frac{y_1}{\cos^2(y_0)}\epsilon$$

$$+ \left(\frac{y_1^2\sin(y_0)}{\cos^3(y_0)} + \frac{y_2}{\cos^2(y_0)}\right)\epsilon^2 + O(\epsilon^3)$$

Substituting the above expansions in the original ODE gives

$$\frac{du_0}{dh} + \frac{du_1}{dh}\epsilon + \frac{du_2}{dh}\epsilon^2 = -\frac{u_0 + u_1\epsilon + u_2\epsilon^2 + O(\epsilon^3)}{(1+h)} - \frac{2Be^{-h/\epsilon}}{\epsilon}$$

$$\times \frac{\left(u_0 + u_1\epsilon + u_2\epsilon^2 + O(\epsilon^3)\right)\left[1 + \lambda\tan\left(y_0 + y_1\epsilon + y_2\epsilon^2 + O(\epsilon^3)\right)\right]}{\epsilon\sin\left(y_0 + y_1\epsilon + y_2\epsilon^2 + O(\epsilon^3)\right)} + O(\epsilon^3)$$

$$= -\frac{u_0 + u_1\epsilon + u_2\epsilon^2 + O(\epsilon^3)}{(1+h)} - 2B\left(u_0 + u_1\epsilon + u_2\epsilon^2 + O(\epsilon^3)\right)$$

$$\times \left[1 + \lambda\left(\tan(y_0) + \frac{y_1}{\cos^2(y_0)}\epsilon + \left(\frac{y_1^2\sin(y_0)}{\cos^3(y_0)} + \frac{y_2}{\cos^2(y_0)}\right)\epsilon^2\right)\right]$$

$$\times \left[\frac{1}{\sin(y_0)} - \frac{y_1\cos(y_0)}{\sin^2(y_0)}\epsilon + \left(\frac{y_1^2\left(1 + \cos^2(y_0)\right)}{2\sin^3(y_0)} - \frac{y_2\cos(y_0)}{\sin^2(y_0)}\right)\epsilon^2\right]$$

29

$$\times \frac{e^{-h/\epsilon}}{\epsilon} + O(\epsilon^3)$$

At this point, all the terms on the right hand side are products of terms which are sums of constant coefficients and powers of ϵ. Thus, multiplying out the above equation and equating terms of identical powers of ϵ, will give the desired expansions (ODEs) of order ϵ. Since $\exp(-1/\epsilon)$ is smaller than any power of ϵ as $\epsilon \to 0$, $\frac{e^{-h/\epsilon}}{\epsilon} \approx 0$ (Nayfeh, 1981:260). This greatly simplifies the expansion above and is frequently used in this study. Thus, the above expansion is multiplied out and the order ϵ terms are as follows

$$\epsilon^0 \text{ terms: } \frac{du_0}{dh} = -\frac{u_0}{(1+h)} \qquad (4.7)$$

$$\epsilon^1 \text{ terms: } \frac{du_1}{dh} = -\frac{u_1}{(1+h)} \qquad (4.8)$$

$$\epsilon^2 \text{ terms: } \frac{du_2}{dh} = -\frac{u_2}{(1+h)} \qquad (4.9)$$

dq/dh Outer Expansion. As derived by using Mathematica™, the dq/dh expansions are grouped by ϵ order as follows:

$$\epsilon^0 \text{ terms: } \frac{dq_0}{dh} = \frac{q_0}{1+h}\left(\frac{q_0^2}{u_0} - 1\right) \qquad (4.10)$$

$$\epsilon^1 \text{ terms: } \frac{dq_1}{dh} = \frac{q_0}{1+h}\left(\frac{2q_0q_1}{u_0} - \frac{q_0^2 u_1}{u_0^2}\right) + \frac{q_1}{1+h}\left(\frac{q_0^2}{u_0} - 1\right) \qquad (4.11)$$

$$\epsilon^2 \text{ terms: } \frac{dq_2}{dh} = q_2\left(\frac{3q_0^2}{u_0} - 1\right) + \frac{3q_0q_1^2}{u_0} - \frac{3q_0^2 q_1 u_1}{u_0^2} + q_0^3\left(\frac{u_1^2}{u_0^3} - \frac{u_2}{u_0^2}\right) \qquad (4.12)$$

30

dI/dh Outer Expansion. As derived by using Mathematica™, the dI/dh expansions are grouped by ϵ order as follows:

ϵ^0 terms: $\quad \dfrac{dI_0}{dh} = 0$ (4.13)

ϵ^1 terms: $\quad \dfrac{dI_1}{dh} = 0$ (4.14)

ϵ^2 terms: $\quad \dfrac{dI_2}{dh} = 0$ (4.15)

dΩ/dh Outer Expansion. As derived by using Mathematica™, the dΩ/dh expansions are grouped by ϵ order as follows:

ϵ^0 terms: $\quad \dfrac{d\Omega_0}{dh} = 0$ (4.16)

ϵ^1 terms: $\quad \dfrac{d\Omega_1}{dh} = 0$ (4.17)

ϵ^2 terms: $\quad \dfrac{d\Omega_2}{dh} = 0$ (4.18)

dα/dh Outer Expansion. As derived by using Mathematica™, the dα/dh expansions are grouped by ϵ order as follows:

ϵ^0 terms: $\quad \dfrac{d\alpha_0}{dh} = \dfrac{1}{(1+h)\tan(\gamma_0)}$ (4.19)

ϵ^1 terms: $\quad \dfrac{d\alpha_1}{dh} = -\dfrac{\gamma_1}{(1+h)\sin^2(\gamma_0)}$ (4.20)

ϵ^2 terms: $\quad \dfrac{d\alpha_2}{dh} = \dfrac{\gamma_1^2\cos(\gamma_0)}{(1+h)\sin^3(\gamma_0)} - \dfrac{\gamma_2}{(1+h)\sin^2(\gamma_0)}$ (4.21)

Outer Expansion Solutions

ϵ^0 Terms. The complete set of ϵ^0 term outer expansion differential equations are derived above and are repeated below.

$$\frac{du_0}{dh} = -\frac{u_0}{(1+h)} \tag{4.7}$$

$$\frac{dq_0}{dh} = \frac{q_0}{1+h}\left(\frac{q_0^2}{u_0} - 1\right) \tag{4.10}$$

$$\frac{dI_0}{dh} = 0 \tag{4.13}$$

$$\frac{d\Omega_0}{dh} = 0 \tag{4.16}$$

$$\frac{d\alpha_0}{dh} = \frac{1}{(1+h)\tan(\gamma_0)} \tag{4.19}$$

Solutions to this set of differential equations are derived in Appendix D and are given below.

$$u_0 = \frac{C_{01}}{1+h} \tag{4.22}$$

$$q_0 = \frac{C_{01}\sqrt{C_{02}}}{\sqrt{1 - \left[C_{01}C_{02}(1+h) - 1\right]^2}} \tag{4.23}$$

$$I_0 = C_{03} \tag{4.24}$$

$$\Omega_0 = C_{04} \tag{4.25}$$

$$\alpha_0 = \text{Sin}^{-1}\left[\frac{1 - \frac{C_{01}}{1+h}}{\sqrt{1 - C_{01}^2 C_{02}}}\right] + C_{05} \tag{4.26}$$

32

$\underline{\epsilon^1 \text{ Terms}}$. The complete set of ϵ^1 term outer expansion differential equations are derived above and are repeated below.

$$\frac{du_1}{dh} = -\frac{u_1}{(1+h)} \tag{4.8}$$

$$\frac{dq_1}{dh} = \frac{q_0}{1+h}\left(\frac{2q_0 q_1}{u_0} - \frac{q_0^2 u_1}{u_0^2}\right) + \frac{q_1}{1+h}\left(\frac{q_0^2}{u_0} - 1\right) \tag{4.11}$$

$$\frac{dI_1}{dh} = 0 \tag{4.14}$$

$$\frac{d\Omega_1}{dh} = 0 \tag{4.17}$$

$$\frac{d\alpha_1}{dh} = -\frac{y_1}{(1+h)\sin^2(y_0)} \tag{4.20}$$

Solutions to this set of differential equations are derived in Appendix D and are given below.

$$u_1 = \frac{C_{11}}{1+h} \tag{4.27}$$

$$q_1 = \frac{\left(\dfrac{C_{11}}{\sqrt{C_{01}(1+h)}} + C_{12}\sqrt{1+h}\right)}{\left(2 - C_{01}C_{02}(1+h)\right)^{\frac{3}{2}}} \tag{4.28}$$

$$I_1 = C_{13} \tag{4.29}$$

$$\Omega_1 = C_{14} \tag{4.30}$$

33

$$\alpha_1 = \frac{\left(C_{12} + C_{11}C_{02}\sqrt{C_{01}}\right)(1+h) - \left(\dfrac{C_{11}}{\sqrt{C_{01}}} + C_{01}C_{12}\right)}{\left(1 - C_{01}^2 C_{02}\right)\sqrt{-C_{01} + 2(1+h) - C_{01}C_{02}(1+h)^2}} + C_{15} \qquad (4.31)$$

$\underline{\epsilon^2 \text{ Terms}}$. The complete set of ϵ^2 term outer expansion differential equations are derived above and are repeated below.

$$\frac{du_2}{dh} = -\frac{u_2}{(1+h)} \qquad (4.9)$$

$$\frac{dq_2}{dh} = q_2\left(\frac{3q_0^2}{u_0} - 1\right) + \frac{3q_0 q_1^2}{u_0} - \frac{3q_0^2 q_1 u_1}{u_0^2} + q_0^3\left(\frac{u_1^2}{u_0^3} - \frac{u_2}{u_0^2}\right) \qquad (4.12)$$

$$\frac{dI_2}{dh} = 0 \qquad (4.15)$$

$$\frac{d\Omega_2}{dh} = 0 \qquad (4.18)$$

$$\frac{d\alpha_2}{dh} = \frac{y_1^2 \cos(y_0)}{(1+h)\sin^3(y_0)} - \frac{y_2}{(1+h)\sin^2(y_0)} \qquad (4.21)$$

Inner Expansions

The solutions developed for the atmospheric (boundary layer) portion of the domain are called the inner solutions. These solutions are developed from asymptotic expansions of the equations of motion using the small parameter, ϵ. The inner solution variables are denoted by the superscript. "i" and are assumed as follows (Vinh and others, 1980:260):

$$u^i = u_0(\xi) + u_1(\xi)\epsilon + u_2(\xi)\epsilon^2 + O(\epsilon^3)$$

34

$$q^i = q_0(\xi) + q_1(\xi)\epsilon + q_2(\xi)\epsilon^2 + O(\epsilon^3)$$

$$I^i = I_0(\xi) + I_1(\xi)\epsilon + I_2(\xi)\epsilon^2 + O(\epsilon^3)$$

$$\Omega^i = \Omega_0(\xi) + \Omega_1(\xi)\epsilon + \Omega_2(\xi)\epsilon^2 + O(\epsilon^3)$$

$$\alpha^i = \alpha_0(\xi) + \alpha_1(\xi)\epsilon + \alpha_2(\xi)\epsilon^2 + O(\epsilon^3)$$

$$\gamma^i = \gamma_0(\xi) + \gamma_1(\xi)\epsilon + \gamma_2(\xi)\epsilon^2 + O(\epsilon^3) \tag{4.32}$$

To derive the inner expansions, a new independent variable is required to force the equations of motion to focus on the boundary layer (Nayfeh, 1981:262). The magnified variable, ξ, becomes the new independent variable for the five coupled equations of motion. Thus the equations of motion undergo a stretching transformation which focuses them on the behavior found in the boundary layer. ξ, the magnified non-dimensional altitude, is defined as

$$\xi = \frac{h}{\epsilon} \quad \text{or} \quad h = \epsilon\,\xi$$

Using the chain rule for differentiation gives an expression for $\frac{d}{d\xi}$.

$$\frac{d}{d\xi} = \frac{dh}{d\xi}\frac{d}{dh} = \epsilon\frac{d}{dh} \quad \rightarrow \quad \frac{d}{dh} = \frac{1}{\epsilon}\frac{d}{d\xi} \tag{4.33}$$

Substituting the above transformations for both h and dh into the original ODEs (Eqs (4.1) - (4.5)) gives the ODEs as functions of the stretched variable, ξ.

$$\frac{du}{d\xi} = -\frac{\epsilon u}{(1+\epsilon\xi)} - \frac{2Bu\big(1 + \lambda\tan(\gamma)\big)}{\sin(\gamma)}e^{-\xi} \tag{4.34}$$

35

$$\frac{dq}{d\xi} = \frac{\epsilon q}{(1 + \epsilon \xi)}\left(\frac{q^2}{u} - 1\right) - B\lambda e^{-\xi} \tag{4.35}$$

$$\frac{dI}{d\xi} = \frac{B\delta \cos(\alpha)}{\sin(\gamma)\cos(\gamma)} e^{-\xi} \tag{4.36}$$

$$\frac{d\Omega}{d\xi} = \frac{B\delta \sin(\alpha)}{\sin(I)\sin(\gamma)\cos(\gamma)} e^{-\xi} \tag{4.37}$$

$$\frac{d\alpha}{d\xi} = \frac{\epsilon}{(1 + \epsilon\xi)\tan(\gamma)} - \frac{B\delta \sin(\alpha)}{\tan(I)\sin(\gamma)\cos(\gamma)} e^{-\xi} \tag{4.38}$$

$du/d\xi$ **Inner Expansion.** This expansion is performed to illustrate the techniques used in deriving an expansion. The original ODE is given in Eq (4.34) as

$$\frac{du}{d\xi} = -\frac{\epsilon u}{(1 + \epsilon\xi)} - \frac{2Bu\left(1 + \lambda \tan(\gamma)\right)}{\sin(\gamma)} e^{-\xi}$$

Next, straightforward expansions for the variables in the above ODE are defined. The expansions for u and γ are assumed in Eq (4.32) while templates for $\tan(\gamma)$, $1/\sin(\gamma)$ and $1/(1 + \epsilon\xi)$ are derived in Appendix C and given as Eqs (C.13), (C.11) and (C.16), respectively. Substituting the expansion for γ in the two trigonometric templates gives

$$\left[\sin\left(\gamma_0 + \gamma_1\epsilon + \gamma_2\epsilon^2 + O(\epsilon^3)\right)\right]^{-1}$$

$$= \frac{1}{\sin(\gamma_0)} - \frac{\gamma_1\cos(\gamma_0)}{\sin^2(\gamma_0)}\epsilon + \left(\frac{\gamma_1^2\left(1 + \cos^2(\gamma_0)\right)}{2\sin^3(\gamma_0)} - \frac{\gamma_2\cos(\gamma_0)}{\sin^2(\gamma_0)}\right)\epsilon^2 + O(\epsilon^3)$$

36

$$\tan\left(\gamma_0 + \gamma_1\epsilon + \gamma_2\epsilon^2 + O(\epsilon^3)\right) = \tan(\gamma_0) + \frac{\gamma_1}{\cos^2(\gamma_0)}\epsilon$$

$$+ \left(\frac{\gamma_1^2\sin(\gamma_0)}{\cos^3(\gamma_0)} + \frac{\gamma_2}{\cos^2(\gamma_0)}\right)\epsilon^2 + O(\epsilon^3)$$

Substituting in the expression $(1 + \epsilon\xi)$ into the algebraic template gives

$$\frac{1}{1 + \epsilon\xi} = 1 - \xi\epsilon + \xi^2\epsilon^2 + O(\epsilon^3)$$

Substituting the above expansions in the original ODE gives

$$\frac{du_0}{d\xi} + \frac{du_1}{d\xi}\epsilon + \frac{du_2}{d\xi}\epsilon^2 = -\epsilon\frac{u_0 + u_1\epsilon + u_2\epsilon^2 + O(\epsilon^3)}{1 + \epsilon\xi} - 2B\,e^{-\xi}$$

$$\times\frac{\left(u_0 + u_1\epsilon + u_2\epsilon^2 + O(\epsilon^3)\right)\left[1 + \lambda\tan\left(\gamma_0 + \gamma_1\epsilon + \gamma_2\epsilon^2 + O(\epsilon^3)\right)\right]}{\sin\left(\gamma_0 + \gamma_1\epsilon + \gamma_2\epsilon^2 + O(\epsilon^3)\right)} + O(\epsilon^3)$$

$$= -\epsilon\left(u_0 + u_1\epsilon + u_2\epsilon^2 + O(\epsilon^3)\right)\left(1 - \xi\epsilon + \xi^2\epsilon^2 + O(\epsilon^3)\right) - 2B\,e^{-\xi}$$

$$\times\left(u_0 + u_1\epsilon + u_2\epsilon^2 + O(\epsilon^3)\right)$$

$$\times\left[1 + \lambda\left(\tan(\gamma_0) + \frac{\gamma_1}{\cos^2(\gamma_0)}\epsilon + \left(\frac{\gamma_1^2\sin(\gamma_0)}{\cos^3(\gamma_0)} + \frac{\gamma_2}{\cos^2(\gamma_0)}\right)\epsilon^2\right)\right]$$

$$\times\left[\frac{1}{\sin(\gamma_0)} - \frac{\gamma_1\cos(\gamma_0)}{\sin^2(\gamma_0)}\epsilon + \left(\frac{\gamma_1^2\left(1 + \cos^2(\gamma_0)\right)}{2\sin^3(\gamma_0)} - \frac{\gamma_2\cos(\gamma_0)}{\sin^2(\gamma_0)}\right)\epsilon^2\right] + O(\epsilon^3)$$

At this point, all the terms on the right hand side are products of terms which are sums of constant coefficients and powers of ϵ. Thus, multiplying out the above equation and equating terms of identical powers of ϵ, will give the desired expansions (ODEs) of order ϵ.

$$\epsilon^0 \text{ terms: } \frac{du_0}{d\xi} = -\frac{2Bu_0 e^{-\xi}\left(1 + \lambda\tan(\gamma_0)\right)}{\sin(\gamma_0)} \tag{4.39}$$

$$\epsilon^1 \text{ terms: } \frac{du_1}{d\xi} = -u_0 - 2Be^{-\xi}\left[\frac{u_1\left(1 + \lambda\tan(\gamma_0)\right)}{\sin(\gamma_0)}\right.$$

$$\left. -\frac{u_0\gamma_1\cos(\gamma_0)\left(1 + \lambda\tan(\gamma_0)\right)}{\sin^2(\gamma_0)} + \frac{\lambda\,u_0\,\gamma_1}{\cos^2(\gamma_0)\,\sin(\gamma_0)}\right] \tag{4.40}$$

$$\epsilon^2 \text{ terms: } \frac{du_2}{d\xi} = -u_1 + u_0\,\xi - 2Be^{-\xi}\left[\frac{\lambda u_0}{\sin(\gamma_0)}\left(\frac{\gamma_2}{\cos^2(\gamma_0)} + \frac{\gamma_1^2\sin(\gamma_0)}{\cos^3(\gamma_0)}\right)\right.$$

$$+ u_0\left(1 + \lambda\tan(\gamma_0)\right)\left[\frac{\gamma_1^2\left(1 + \cos^2(\gamma_0)\right)}{2\sin^3(\gamma_0)} - \frac{\gamma_2\cos(\gamma_0)}{\sin^2(\gamma_0)}\right]$$

$$+ \frac{\lambda u_1\gamma_1}{\cos^2(\gamma_0)\,\sin(\gamma_0)} - \frac{u_1\gamma_1\cos(\gamma_0)\left(1 + \lambda\tan(\gamma_0)\right)}{\sin^2(\gamma_0)}$$

$$\left. -\frac{\lambda u_0\gamma_1^2}{\sin^2(\gamma_0)\,\cos(\gamma_0)} + \frac{u_2\left(1 + \lambda\tan(\gamma_0)\right)}{\sin(\gamma_0)}\right] \tag{4.41}$$

$\underline{dq/d\xi \text{ Inner Expansion}}$. As derived by using Mathematica™, the dq/dh expansions are grouped by ϵ order as follows:

38

ϵ^0 terms:
$$\frac{dq_0}{d\xi} = -\lambda B e^{-\xi} \tag{4.42}$$

ϵ^1 terms:
$$\frac{dq_1}{d\xi} = q_0\left(-1 + \frac{q_0^2}{u_0}\right) \tag{4.43}$$

ϵ^2 terms:
$$\frac{dq_2}{d\xi} = \frac{q_0^3}{u_0^2}\left(1 - u_1 - \xi u_0\right) + q_0(\xi - 1) + q_1\left(\frac{q_0^2}{u_0} - 1\right) + \frac{2q_0 q_1 q_2}{u_0} \tag{4.44}$$

dI/dξ Inner Expansion. As derived by using Mathematica™, the dI/dh expansions are grouped by ϵ order as follows:

ϵ^0 terms:
$$\frac{dI_0}{d\xi} = \frac{B\delta e^{-\xi}\cos(\alpha_0)}{\sin(\gamma_0)\cos(\gamma_0)} \tag{4.45}$$

ϵ^1 terms:
$$\frac{dI_1}{d\xi} = B\delta e^{-\xi}\left[\gamma_1\cos(\alpha_0)\left(\frac{1}{\cos^2(\gamma_0)} - \frac{1}{\sin^2(\gamma_0)}\right)\right.$$
$$\left. - \frac{\alpha_1\sin(\alpha_0)}{\sin(\gamma_0)\cos(\gamma_0)}\right] \tag{4.46}$$

ϵ^2 terms:
$$\frac{dI_2}{d\xi} = B\delta e^{-\xi}\left\{\left(\gamma_2\cos(\alpha_0) - \gamma_1\alpha_1\sin(\alpha_0)\right)\left(\frac{1}{\cos^2(\gamma_0)} - \frac{1}{\sin^2(\gamma_0)}\right)\right.$$
$$+ \frac{\gamma_1^2\cos(\alpha_0)}{\sin(\gamma_0)\cos(\gamma_0)}\left(\frac{1}{\cos^2(\gamma_0)} - \frac{1}{\sin^2(\gamma_0)} - 1\right)$$
$$\left. - \frac{\left(\frac{\alpha_1^2}{2} + \gamma_1^2\right)\cos(\alpha_0) + \alpha_2\sin(\alpha_0)}{\sin(\gamma_0)\cos(\gamma_0)}\right\} \tag{4.47}$$

<u>$d\Omega/d\xi$ Inner Expansion</u>. As derived by using Mathematica™, the $d\Omega/dh$ expansions are grouped by ϵ order as follows:

ϵ^0 terms:
$$\frac{d\Omega_0}{d\xi} = \frac{B\delta e^{-\xi} \sin(\alpha_0)}{\sin(I_0)\sin(\gamma_0)\cos(\gamma_0)} \qquad (4.48)$$

ϵ^1 terms:
$$\frac{d\Omega_1}{d\xi} = B\delta e^{-\xi}\left[\frac{\gamma_1 \sin(\alpha_0)}{\sin(I_0)}\left(\frac{1}{\cos^2(\gamma_0)} - \frac{1}{\sin^2(\gamma_0)}\right)\right.$$

$$\left. - \frac{I_1 \sin(\alpha_0)\cos(I_0)}{\sin^2(I_0)\sin(\gamma_0)\cos(\gamma_0)} + \frac{\alpha_1\cos(\alpha_0)}{\sin(I_0)\sin(\gamma_0)\cos(\gamma_0)}\right] \quad (4.49)$$

ϵ^2 terms:
$$\frac{d\Omega_2}{d\xi} = B\delta e^{-\xi}\left\{\left(\frac{1}{\cos^2(\gamma_0)} - \frac{1}{\sin^2(\gamma_0)}\right)\left[\frac{\gamma_1\alpha_1\cos(\alpha_0)}{\sin(I_0)}\right.\right.$$

$$\left.+ \frac{\gamma_1^2}{\sin(\gamma_0)\cos(\gamma_0)\sin(\alpha_0)\sin(I_0)} + \frac{\gamma_2}{\sin(\alpha_0)\sin(I_0)} - \frac{\gamma_1 I_1 \sin(\alpha_0)\cos(I_0)}{\sin^2(I_0)}\right]$$

$$+ \frac{1}{\sin(\gamma_0)\cos(\gamma_0)\sin(\alpha_0)\sin(I_0)}\left[\frac{I_1^2\left(1 + \cos^2(I_0)\right)}{2\sin^2(I_0)} + \alpha_2\sin(\alpha_0)\cos(\alpha_0)\right.$$

$$- \frac{\alpha_1^2\sin^2(\alpha_0)}{2} - \gamma_1^2\left(1 + \sin^2(\alpha_0)\right)$$

$$\left.\left.- \frac{\cos(I_0)}{\sin(I_0)}\left(I_2 + \alpha_1 I_1 \sin(\alpha_0)\cos(\alpha_0)\right)\right]\right\} \qquad (4.50)$$

<u>$d\alpha/d\xi$ Inner Expansion</u>. As derived by using Mathematica™, the $d\alpha/dh$ expansions are grouped by ϵ order as follows:

40

ϵ^0 terms:
$$\frac{d\alpha_0}{d\xi} = -\frac{B\delta e^{-\xi}\sin(\alpha_0)}{\tan(I_0)\sin(\gamma_0)\cos(\gamma_0)} \tag{4.51}$$

ϵ^1 terms:
$$\frac{d\alpha_1}{d\xi} = \frac{1}{\tan(\gamma_0)} - B\delta e^{-\xi}\left[\frac{\gamma_1\sin(\alpha_0)}{\tan(I_0)}\left(\frac{1}{\cos^2(\gamma_0)} - \frac{1}{\sin^2(\gamma_0)}\right)\right.$$

$$\left. - \frac{I_1\sin(\alpha_0)}{\sin^2(I_0)\sin(\gamma_0)\cos(\gamma_0)} + \frac{\alpha_1\cos(\alpha_0)}{\tan(I_0)\sin(\gamma_0)\cos(\gamma_0)}\right] \tag{4.52}$$

ϵ^2 terms:
$$\frac{d\alpha_2}{d\xi} = -\frac{\gamma_1}{\sin^2(\gamma_0)} - \frac{\xi}{\tan(\gamma_0)} - B\delta e^{-\xi}\left\{\left(\frac{1}{\cos^2(\gamma_0)} - \frac{1}{\sin^2(\gamma_0)}\right)\left[\frac{\gamma_1\alpha_1\cos(\alpha_0)}{\tan(I_0)}\right.\right.$$

$$\left. + \frac{\gamma_1{}^2}{\sin(\gamma_0)\cos(\gamma_0)\sin(\alpha_0)\tan(I_0)} + \frac{\gamma_2}{\sin(\alpha_0)\tan(I_0)} - \frac{\gamma_1 I_1\sin(\alpha_0)}{\sin^2(I_0)}\right]$$

$$+ \frac{1}{\sin(\gamma_0)\cos(\gamma_0)\sin(\alpha_0)\tan(I_0)}\left[\frac{I_1{}^2\cos(I_0)}{2\sin^2(I_0)} - \gamma_1{}^2\left(1 + \sin^2(\alpha)\right)\right.$$

$$\left.\left. + \cos(I_0)\cos(\alpha_0)\left(\alpha_2\cos(\alpha_0) - \alpha_1{}^2\sin(\alpha_0)\right) - \frac{I_2 + \alpha_1 I_1\sin(\alpha_0)\cos(\alpha_0)}{\sin(I_0)}\right]\right\} \tag{4.53}$$

Inner Expansion Solutions

$\underline{\epsilon^0 \text{ Terms.}}$ The complete set of ϵ^0 term inner expansion differential equations are derived above and are repeated below.

$$\frac{du_0}{d\xi} = -\frac{2Bu_0 e^{-\xi}\left(1 + \lambda\tan(\gamma_0)\right)}{\sin(\gamma_0)} \tag{4.39}$$

41

$$\frac{dq_0}{d\xi} = -\lambda Be^{-\xi} \tag{4.42}$$

$$\frac{dI_0}{d\xi} = \frac{B\delta e^{-\xi}\cos(\alpha_0)}{\sin(\gamma_0)\cos(\gamma_0)} \tag{4.45}$$

$$\frac{d\Omega_0}{d\xi} = \frac{B\delta e^{-\xi}\sin(\alpha_0)}{\sin(I_0)\sin(\gamma_0)\cos(\gamma_0)} \tag{4.48}$$

$$\frac{d\alpha_0}{d\xi} = -\frac{B\delta e^{-\xi}\sin(\alpha_0)}{\tan(I_0)\sin(\gamma_0)\cos(\gamma_0)} \tag{4.51}$$

Solutions to this set of differential equations are derived in Appendix E and are given below.

$$u_0 = K_{01}\left(\lambda Be^{-\xi} + K_{02}\right)^2 \exp\left[-\frac{2}{\lambda}\cos^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)\right] \tag{4.54}$$

$$q_0 = \lambda Be^{-\xi} + K_{02} \tag{4.55}$$

$$I_0 = \cos^{-1}\left\{\cos(K_{03})\cos\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4} + \frac{\cos^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)}{2}\right)\right) + K_{05}\right]\right\} \tag{4.56}$$

$$\Omega_0 = K_{04} - \cos^{-1}\left(\frac{\cos(\alpha_0)}{\cos(K_{03})}\right) = K_{04} \tag{4.57}$$

$$-\cos^{-1}\left(\frac{\left\{1 - \left[\dfrac{\sin^2(K_{03})}{1 - \cos^2(K_{03})\cos^2\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4} + \frac{\cos^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)}{2}\right)\right) + K_{05}\right]}\right]^2\right\}^{\frac{1}{2}}}{\cos(K_{03})}\right)$$

$$\alpha_0 = \sin^{-1}\left(\frac{\sin(K_{03})}{\sin(I_0)}\right) \tag{4.58}$$

42

$$- \text{Sin}^{-1}\left[\cfrac{\sin(K_{03})}{\left[1-\cos^2(K_{03})\cos^2\left[\cfrac{\delta}{\lambda}\ln\left(\tan\left(\cfrac{\pi}{4}+\cfrac{\text{Cos}^{-1}\left(\lambda Be^{-\xi}+K_{02}\right)}{2}\right)\right)+K_{05}\right]\right]^{1/2}}\right]$$

$\underline{\epsilon^1\ \text{Terms}}$. The complete set of ϵ^1 term inner expansion differential equations are derived above and are repeated below.

$$\frac{du_1}{d\xi} = -u_0 - 2Be^{-\xi}\left[\frac{u_1\left(1+\lambda\tan(\gamma_0)\right)}{\sin(\gamma_0)}\right.$$

$$\left. - \frac{u_0\gamma_1\cos(\gamma_0)\left(1+\lambda\tan(\gamma_0)\right)}{\sin^2(\gamma_0)} + \frac{\lambda\ u_0\ \gamma_1}{\cos^2(\gamma_0)\ \sin(\gamma_0)}\right] \qquad (4.40)$$

$$\frac{dq_1}{d\xi} = q_0\left(-1+\frac{q_0^2}{u_0}\right) \qquad (4.43)$$

$$\frac{dI_1}{d\xi} = B\delta e^{-\xi}\left[\gamma_1\cos(\alpha_0)\left(\frac{1}{\cos^2(\gamma_0)}-\frac{1}{\sin^2(\gamma_0)}\right)-\frac{\alpha_1\sin(\alpha_0)}{\sin(\gamma_0)\cos(\gamma_0)}\right] \qquad (4.46)$$

$$\frac{d\Omega_1}{d\xi} = B\delta e^{-\xi}\left[\frac{\gamma_1\sin(\alpha_0)}{\sin(I_0)}\left(\frac{1}{\cos^2(\gamma_0)}-\frac{1}{\sin^2(\gamma_0)}\right)\right.$$

$$\left. - \frac{I_1\ \sin(\alpha_0)\ \cos(I_0)}{\sin^2(I_0)\ \sin(\gamma_0)\ \cos(\gamma_0)} + \frac{\alpha_1\cos(\alpha_0)}{\sin(I_0)\ \sin(\gamma_0)\ \cos(\gamma_0)}\right] (4.49)$$

43

$$\frac{d\alpha_1}{d\xi} = \frac{1}{\tan(\gamma_0)} - B\delta e^{-\xi}\left[\frac{\gamma_1 \sin(\alpha_0)}{\tan(I_0)}\left(\frac{1}{\cos^2(\gamma_0)} - \frac{1}{\sin^2(\gamma_0)}\right)\right.$$

$$\left. - \frac{I_1 \sin(\alpha_0)}{\sin^2(I_0)\sin(\gamma_0)\cos(\gamma_0)} + \frac{\alpha_1\cos(\alpha_0)}{\tan(I_0)\sin(\gamma_0)\cos(\gamma_0)}\right] \quad (4.52)$$

Solutions to this set of differential equations are derived in Appendix E and are given below.

$$u_1 = \left(\lambda Be^{-\xi} + K_{02}\right)^2 \exp\left(-\frac{2}{\lambda}\text{Cos}^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)\right)$$

$$\times \left\{K_{11} + \frac{2K_{01}\left(K_{12} + K_{02}K_{121}\right)(\xi + 1)}{\lambda Be^{-\xi} + K_{02}} - \frac{2K_{121}K_{01}}{\lambda}\text{Sin}^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)\right.$$

$$+ \frac{\frac{2}{\lambda}\left[K_{01}K_{12} - K_{01}K_{121}\right]\left(K_{02}\xi - \lambda Be^{-\xi}\right)}{\sqrt{1 - \left(\lambda Be^{-\xi} + K_{02}\right)^2}} + \frac{2K_{02}K_{01}K_{121}}{\lambda\sqrt{1 - K_{02}^2}}$$

$$\left.\times \ln\left[\frac{2\sqrt{(1-K_{02}^2)\left[1-(\lambda Be^{-\xi}+K_{02})^2\right]} - 2K_{02}\left(\lambda Be^{-\xi} + K_{02}\right) + 2}{\lambda Be^{-\xi}}\right]\right\} \quad (4.59)$$

$$q_1 = K_{121}\left(K_{02}\xi - \lambda Be^{-\xi}\right) + K_{12} \quad (4.60)$$

$$I_1 = K_{13} + \frac{\delta}{\lambda}\left\{\frac{K_{131}\left(K_{132} - \frac{K_{133}}{\lambda Be^{-\xi} + K_{02}}\right)}{\sqrt{1 - \left(\lambda Be^{-\xi} + K_{02}\right)^2}}\right.$$

$$+ \left[K_{131}K_{132} + K_{134}\left(\frac{\pi}{2} - K_{15}\right)\right]\ln\left[\tan\left(\frac{\pi}{4} + \frac{\text{Cos}^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)}{2}\right)\right]$$

$$+ K_{134}\left[\frac{K_{02}\cos^{-1}\left(\lambda Be^{-\xi}+K_{02}\right)}{\sqrt{1-K_{02}^2}}-\frac{\left[\cos^{-1}\left(\lambda Be^{-\xi}+K_{02}\right)\right]^2}{2}\right.$$

$$\left.\left.-\frac{\left[\cos^{-1}\left(\lambda Be^{-\xi}+K_{02}\right)\right]^4}{8}\right]\right] \tag{4.61}$$

$$\Omega_1 = K_{14}+\frac{\delta}{\lambda}\left\{\frac{K_{141}\left(K_{132}-\dfrac{K_{133}}{\lambda Be^{-\xi}+K_{02}}\right)}{\sqrt{1-\left(\lambda Be^{-\xi}+K_{02}\right)^2}}+\frac{K_{142}K_{133}}{\lambda Be^{-\xi}+K_{02}}\right.$$

$$+ K_{143}\ln\left[\tan\left(\frac{\pi}{4}+\frac{\cos^{-1}\left(\lambda Be^{-\xi}+K_{02}\right)}{2}\right)\right]$$

$$+\left(K_{132}K_{142}+K_{144}\left(\frac{\pi}{2}-K_{15}\right)\right)\left[\frac{4\ln\left(\lambda Be^{-\xi}+K_{02}\right)}{3}-\frac{\left(\lambda Be^{-\xi}+K_{02}\right)^2}{6}\right]$$

$$+\frac{K_{02}K_{144}\cos^{-1}\left(\lambda Be^{-\xi}+K_{02}\right)}{\sqrt{1-K_{02}^2}}$$

$$+ K_{145}\left[\frac{\left[\cos^{-1}\left(\lambda Be^{-\xi}+K_{02}\right)\right]^3}{6}+\frac{3\left[\cos^{-1}\left(\lambda Be^{-\xi}+K_{02}\right)\right]^5}{40}\right]$$

$$\left.+ K_{146}\left[\frac{\left[\cos^{-1}\left(\lambda Be^{-\xi}+K_{02}\right)\right]^2}{2}+\frac{\left[\cos^{-1}\left(\lambda Be^{-\xi}+K_{02}\right)\right]^4}{8}\right]\right\} \tag{4.62}$$

$$\alpha_1 = K_{15}-\sin^{-1}\left(\lambda Be^{-\xi}+K_{02}\right)+\frac{K_{02}}{\sqrt{1-K_{02}^2}}$$

$$\times\ln\left[\frac{2\sqrt{\left(1-K_{02}^2\right)\left[1-\left(\lambda Be^{-\xi}+K_{02}\right)^2\right]}-2K_{02}\left(\lambda Be^{-\xi}+K_{02}\right)+2}{\lambda Be^{-\xi}}\right] \tag{4.63}$$

$\underline{\epsilon^2 \text{ Terms}}$. The complete set of ϵ^2 term inner expansion differential equations are derived above and are repeated below.

$$\frac{du_2}{d\xi} = -u_1 + u_0\,\xi - 2Be^{-\xi}\left[\frac{\lambda u_0}{\sin(\gamma_0)}\left(\frac{\gamma_2}{\cos^2(\gamma_0)} + \frac{\gamma_1^2\sin(\gamma_0)}{\cos^3(\gamma_0)}\right)\right.$$

$$+ u_0\left(1 + \lambda\tan(\gamma_0)\right)\left[\frac{\gamma_1^2\left(1 + \cos^2(\gamma_0)\right)}{2\sin^3(\gamma_0)} - \frac{\gamma_2\cos(\gamma_0)}{\sin^2(\gamma_0)}\right]$$

$$+ \frac{\lambda u_1\gamma_1}{\cos^2(\gamma_0)\sin(\gamma_0)} - \frac{u_1\gamma_1\cos(\gamma_0)\left(1 + \lambda\tan(\gamma_0)\right)}{\sin^2(\gamma_0)}$$

$$\left. - \frac{\lambda u_0\gamma_1^2}{\sin^2(\gamma_0)\cos(\gamma_0)} + \frac{u_2\left(1 + \lambda\tan(\gamma_0)\right)}{\sin(\gamma_0)}\right] \qquad (4.41)$$

$$\frac{dq_2}{d\xi} = \frac{q_0^3}{u_0^2}\left(1 - u_1 - \xi\,u_0\right) + q_0(\xi - 1) + q_1\left(\frac{q_0^2}{u_0} - 1\right) + \frac{2q_0 q_1 q_2}{u_0} \qquad (4.44)$$

$$\frac{dI_2}{d\xi} = B\delta e^{-\xi}\left\{\left(\gamma_2\cos(\alpha_0) - \gamma_1\alpha_1\sin(\alpha_0)\right)\left(\frac{1}{\cos^2(\gamma_0)} - \frac{1}{\sin^2(\gamma_0)}\right)\right.$$

$$+ \frac{\gamma_1^2\cos(\alpha_0)}{\sin(\gamma_0)\cos(\gamma_0)}\left(\frac{1}{\cos^2(\gamma_0)} - \frac{1}{\sin^2(\gamma_0)} - 1\right)$$

$$\left. - \frac{\left(\frac{\alpha_1^2}{2} + \gamma_1^2\right)\cos(\alpha_0) + \alpha_2\sin(\alpha_0)}{\sin(\gamma_0)\cos(\gamma_0)}\right\} \qquad (4.47)$$

46

$$\frac{d\Omega_2}{d\xi} = B\delta e^{-\xi} \left\{ \left(\frac{1}{\cos^2(\gamma_0)} - \frac{1}{\sin^2(\gamma_0)} \right) \left[\frac{\gamma_1 \alpha_1 \cos(\alpha_0)}{\sin(I_0)} \right. \right.$$

$$+ \frac{\gamma_1^2}{\sin(\gamma_0)\cos(\gamma_0)\sin(\alpha_0)\sin(I_0)} + \frac{\gamma_2}{\sin(\alpha_0)\sin(I_0)} - \frac{\gamma_1 I_1 \sin(\alpha_0)\cos(I_0)}{\sin^2(I_0)} \right]$$

$$+ \frac{1}{\sin(\gamma_0)\cos(\gamma_0)\sin(\alpha_0)\sin(I_0)} \left[\frac{I_1^2 \left(1 + \cos^2(I_0) \right)}{2\sin^2(I_0)} + \alpha_2 \sin(\alpha_0)\cos(\alpha_0) \right.$$

$$- \frac{\alpha_1^2 \sin^2(\alpha_0)}{2} - \gamma_1^2 \left(1 + \sin^2(\alpha_0) \right)$$

$$\left. \left. - \frac{\cos(I_0)}{\sin(I_0)} \left(I_2 + \alpha_1 I_1 \sin(\alpha_0)\cos(\alpha_0) \right) \right] \right\} \qquad (4.50)$$

$$\frac{d\alpha_2}{d\xi} = - \frac{\gamma_1}{\sin^2(\gamma_0)} - \frac{\xi}{\tan(\gamma_0)} - B\delta e^{-\xi} \left\{ \left(\frac{1}{\cos^2(\gamma_0)} - \frac{1}{\sin^2(\gamma_0)} \right) \left[\frac{\gamma_1 \alpha_1 \cos(\alpha_0)}{\tan(I_0)} \right. \right.$$

$$+ \frac{\gamma_1^2}{\sin(\gamma_0)\cos(\gamma_0)\sin(\alpha_0)\tan(I_0)} + \frac{\gamma_2}{\sin(\alpha_0)\tan(I_0)} - \frac{\gamma_1 I_1 \sin(\alpha_0)}{\sin^2(I_0)} \right]$$

$$+ \frac{1}{\sin(\gamma_0)\cos(\gamma_0)\sin(\alpha_0)\tan(I_0)} \left[\frac{I_1^2 \cos(I_0)}{2\sin^2(I_0)} - \gamma_1^2 \left(1 + \sin^2(\alpha) \right) \right.$$

$$\left. \left. + \cos(I_0)\cos(\alpha_0) \left(\alpha_2 \cos(\alpha_0) - \alpha_1^2 \sin(\alpha_0) \right) - \frac{I_2 + \alpha_1 I_1 \sin(\alpha_0)\cos(\alpha_0)}{\sin(I_0)} \right] \right\} (4.53)$$

Matching Asymptotic Expansions

47

Once the asymptotic expansions for the equations of motion are derived, the outer and inner solutions are 'matched' to reduce the number of unknowns, the constants of integration, resulting in an equal number of initial conditions and constants of integration. Thus, the final solutions to the equations of motion, known as the composite solutions, are used to solve entry trajectories.

The underlying idea of using outer and inner solutions is to derive two solutions which model the two dominant forces acting on the entry vehicle, gravity and air drag. To model each of these forces, two scales, the independent variables, are used to derive two separate expansions; each are valid in part of the domain of the entry problem, but neither covers the entire domain of interest. Although each of the scales do not cover the entire domain, they do overlap, or have regions where both expansions are valid. Since the expansions having neighboring regions of validity, they are blended or matched, resulting in one composite solution which connects the two previously separate solutions.

In this study, Van Dyke's matching principle (Nayfeh, 1981:282-283) is used to equate the outer and inner constants of integration. Van Dyke's principle states that the mth inner expansion of the nth outer expansion equals the nth outer expansion of the mth inner expansion, where m,n are integer values. Simply stated, where y is the derived expansion and i and o are the outer and inner expansions

$$\left[(y)^o\right]^i = \left[(y)^i\right]^o \tag{4.64}$$

To obtain the inner expansion of the outer expansion, the outer expansion is rewritten in terms of the inner variable, ξ, and is expanded for small ϵ, keeping ξ constant. To obtain the outer expansion of the inner expansion, the inner expansion is rewritten in terms of the outer variable, h, and is expanded for small ϵ, keeping h constant. The ϵ^0 terms of the two resulting expressions are equated, thus expressing the inner constants of integration, K_{ij}, in terms of the outer constants of integration, C_{ij}, correct to $O(\epsilon)$.

Zero Order ϵ Matching

In the following sections, Van Dyke's matching principle is applied to the ϵ^0 inner and outer expansion solutions. The outer expansions will be rewritten in terms of the inner variable first, and then the inner expansions will be rewritten in terms of the outer variable. Both composite expansions will be expanded for ϵ and their resulting ϵ^0 terms will be equated, resulting in the inner expansion constants of integration being expressed in terms of the outer expansion constants of integration.

Inner Expansion of the Outer Expansion Solutions. In this section, the five outer expansion solutions are rewritten in terms of the inner variable and expanded to obtain ϵ^0 terms.

Matching Expansion for u_0. From Eq (4.22), the ϵ^0 outer expansion solution for u is

$$\left(u_0\right)^0 = \frac{C_{01}}{1+h}$$

Rewriting the outer variable, h, in terms of the inner variable, ξ, uses the definition $h = \epsilon \xi$. Substituting this in the above equation gives

$$\left(u_0\right)^o = \frac{C_{01}}{1 + \epsilon \xi}$$

Using the binomial expansion derived as Eq (C.16) gives

$$\frac{1}{1 + \epsilon \xi} = 1 - \xi \epsilon + \xi^2 \epsilon^2 + O(\epsilon^3) = 1 + O(\epsilon)$$

Thus, the inner expansion of the ϵ^0 outer expansion solution for u is

$$\left[\left(u_0\right)^o\right]^i = C_{01} \tag{4.65}$$

$\underline{\text{Matching Expansion for } q_0}$. From Eq (4.23), the ϵ^0 outer expansion solution for q is

$$\left(q_0\right)^o = \frac{C_{01}\sqrt{C_{02}}}{\sqrt{1 - \left[C_{01}C_{02}(1 + h) - 1\right]^2}} = \sqrt{\frac{C_{01}}{2(1 + h) - C_{01} C_{02}(1 + h)^2}}$$

Rewriting the outer variable, h, in terms of the inner variable, ξ, uses the definition $h = \epsilon \xi$. Substituting this in the above equation gives

$$\left(q_0\right)^o = \frac{\sqrt{C_{01}}}{\left(-C_{01} C_{02}\epsilon^2 \xi^2 + 2(1 - C_{01} C_{02})\epsilon \xi + (2 - C_{01} C_{02})\right)^{1/2}}$$

Using the binomial expansion derived as Eq (C.17) gives

$$\frac{1}{\left(-C_{01}C_{02}\epsilon^2\xi^2 + 2(1 - C_{01}C_{02})\epsilon\xi + (2 - C_{01}C_{02})\right)^{1/2}} = \frac{1}{(2 - C_{01}C_{02})^{1/2}} + O(\epsilon)$$

50

Thus, the inner expansion of the ϵ^0 outer expansion solution for q is

$$\left[(q_0)^0\right]^i = \sqrt{\frac{C_{01}}{2 - C_{01} C_{02}}} \qquad (4.66)$$

Matching Expansion for I_0. From Eq (4.24), the ϵ^0 outer expansion solution for I is

$$(I_0)^0 = C_{03}$$

By inspection, since there is no dependence on h, the inner expansion of the ϵ^0 outer expansion solution for I is

$$\left[(I_0)^0\right]^i = C_{03} \qquad (4.67)$$

Matching Expansion for Ω_0. From Eq (4.25), the ϵ^0 outer expansion solution for Ω is

$$(\Omega_0)^0 = C_{04}$$

By inspection, since there is no dependence on h, the inner expansion of the ϵ^0 outer expansion solution for Ω is

$$\left[(\Omega_0)^0\right]^i = C_{04} \qquad (4.68)$$

Matching Expansion for α_0.. From Eq (4.26), the ϵ^0 outer expansion solution for α is

$$(\alpha_0)^0 = \text{Sin}^{-1}\left[\frac{1 - \dfrac{C_{01}}{1 + h}}{\sqrt{1 - C_{01}^2 C_{02}}}\right] + C_{05}$$

Rewriting the outer variable, h, in terms of the inner variable, ξ, uses the definition $h = \epsilon \xi$. Substituting this in the above equation gives

$$\left(\alpha_0\right)^o = \text{Sin}^{-1}\left[\frac{1 - \dfrac{C_{01}}{1 + \epsilon\xi}}{\sqrt{1 - C_{01}{}^2 C_{02}}}\right] + C_{05}$$

Using the binomial expansion derived as Eq (C.16) gives

$$\frac{1}{1 + \epsilon\xi} = 1 + O(\epsilon)$$

Thus, the inner expansion of the ϵ^0 outer expansion solution for α is

$$\left[\left(\alpha_0\right)^o\right]^i = \text{Sin}^{-1}\left[\frac{1 - C_{01}}{\sqrt{1 - C_{01}{}^2 C_{02}}}\right] + C_{05} \tag{4.69}$$

<u>Outer Expansion of the Inner Expansion Solutions</u>. In this section, the five inner expansion solutions are rewritten in terms of the outer variable and expanded to obtain ϵ^0 terms.

<u>Matching Expansion for u_0</u>. From Eq (4.54), the ϵ^0 inner expansion solution for u is

$$\left(u_0\right)^i = K_{01}\left(\lambda B e^{-\xi} + K_{02}\right)^2 \exp\left[-\frac{2}{\lambda}\text{Cos}^{-1}\left(\lambda B e^{-\xi} + K_{02}\right)\right]$$

Rewriting the inner variable, ξ, in terms of the outer variable, h, uses the definition $\xi = h/\epsilon$. Substituting this in the above equation gives

$$\left(u_0\right)^i = K_{01}\left(\lambda B e^{-h/\epsilon} + K_{02}\right)^2 \exp\left[-\frac{2}{\lambda}\text{Cos}^{-1}\left(\lambda B e^{-h/\epsilon} + K_{02}\right)\right]$$

52

Since $\exp(-1/\epsilon)$ decreases to zero rapidly as $\epsilon \to 0$, $e^{-h/\epsilon} = 0$ (Nayfeh, 1981:260). Therefore

$$\lambda B e^{-h/\epsilon} + K_{02} = K_{02}$$

Thus, the outer expansion of the ϵ^0 inner expansion solution for u is

$$\left[(u_0)^i \right]^0 = K_{01} K_{02}^{2} \exp\left[-\frac{2}{\lambda} \cos^{-1}(K_{02}) \right] \tag{4.70}$$

Matching Expansion for q_0. From Eq (4.55), the ϵ^0 inner expansion solution for q is

$$\left(q_0 \right)^i = \lambda B e^{-\xi} + K_{02}$$

Rewriting the inner variable, ξ, in terms of the outer variable, h, uses the definition $\xi = h/\epsilon$. Substituting this in the above equation gives

$$\left(q_0 \right)^i = \lambda B e^{-h/\epsilon} + K_{02}$$

Since $\exp(-1/\epsilon)$ decreases to zero rapidly as $\epsilon \to 0$, $e^{-h/\epsilon} = 0$ (Nayfeh, 1981:260). Therefore

$$\lambda B e^{-h/\epsilon} + K_{02} = K_{02}$$

Thus, the outer expansion of the ϵ^0 inner expansion solution for q is

$$\left[(q_0)^i \right]^0 = K_{02} \tag{4.71}$$

Matching Expansion for I_0. From Eq (4.56), the ϵ^0 inner expansion solution for I is

$$\left(I_0\right)^i = Cos^{-1}\left\{\cos(K_{03})\cos\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4}+\frac{Cos^{-1}\left(\lambda Be^{-\xi}+K_{02}\right)}{2}\right)\right)\right]+K_{05}\right\}$$

Rewriting the inner variable, ξ, in terms of the outer variable, h, uses the definition $\xi = h/\epsilon$. Substituting this in the above equation gives

$$\left(I_0\right)^i = Cos^{-1}\left\{\cos(K_{03})\cos\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4}+\frac{Cos^{-1}\left(\lambda Be^{-h/\epsilon}+K_{02}\right)}{2}\right)\right)\right]+K_{05}\right\}$$

Since $\exp(-1/\epsilon)$ decreases to zero rapidly as $\epsilon\to 0$, $e^{-h/\epsilon} = 0$. Therefore

$$\lambda Be^{-h/\epsilon}+K_{02} = K_{02}$$

Thus, the outer expansion of the ϵ^0 inner expansion solution for I is

$$\left[\left(I_0\right)^i\right]^o = Cos^{-1}\left\{\cos(K_{03})\cos\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4}+\frac{Cos^{-1}(K_{02})}{2}\right)\right)\right]+K_{05}\right\} \quad (4.72)$$

Matching Expansion for α_0. From Eq (4.58), the ϵ^0 inner expansion solution for α is

$$\left(\alpha_0\right)^i = Sin^{-1}\left(\frac{\sin(K_{03})}{\sin\left(\left(I_0\right)^i\right)}\right)$$

$$= Sin^{-1}\left[\frac{\sin(K_{03})}{\left[\left[1-\cos^2(K_{03})\cos^2\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4}+\frac{Cos^{-1}\left(\lambda Be^{-\xi}+K_{02}\right)}{2}\right)\right)\right]+K_{05}\right]\right]^{1/2}}\right]$$

Rewriting the inner variable, ξ, in terms of the outer variable, h, uses the definition $\xi = h/\epsilon$. Substituting this in the above equation gives

54

$$(\alpha_0)^i = \operatorname{Sin}^{-1}\left[\frac{\sin(K_{03})}{\left[\left[1-\cos^2(K_{03})\cos^2\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4}+\frac{\operatorname{Cos}^{-1}\left(\lambda Be^{-h/\epsilon}+K_{02}\right)}{2}\right)\right)\right]+K_{05}\right]\right]^{1/2}}\right]$$

Since $\exp(-1/\epsilon)$ decreases to zero rapidly as $\epsilon \to 0$, $e^{-h/\epsilon} = 0$. Therefore

$$\lambda Be^{-h/\epsilon} + K_{02} = K_{02}$$

Thus, the outer expansion of the ϵ^0 inner expansion solution for α is

$$\left[(\alpha_0)^i\right]^0 = \operatorname{Sin}^{-1}\left(\frac{\sin(K_{03})}{\sin\left\{\left[(I_0)^i\right]^0\right\}}\right) \tag{4.73}$$

$$= \operatorname{Sin}^{-1}\left[\frac{\sin(K_{03})}{\left[\left[1-\cos^2(K_{03})\cos^2\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4}+\frac{\operatorname{Cos}^{-1}(K_{02})}{2}\right)\right)\right]+K_{05}\right]\right]^{1/2}}\right]$$

<u>Matching Expansion for Ω_0</u>. From Eq (4.57), the ϵ^0 inner expansion solution for Ω is

$$(\Omega_0)^i = K_{04} - \operatorname{Cos}^{-1}\left(\frac{\cos\left((\alpha_0)^i\right)}{\cos(K_{03})}\right) = K_{04}$$

$$- \operatorname{Cos}^{-1}\left(\frac{\left\{\left[1-\left[\frac{\sin^2(K_{03})}{\left[1-\cos^2(K_{03})\cos^2\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4}+\frac{\operatorname{Cos}^{-1}\left(\lambda Be^{-\xi}+K_{02}\right)}{2}\right)\right)\right]+K_{05}\right]}\right]^2\right]\right\}^{1/2}}{\cos(K_{03})}\right)$$

Rewriting the inner variable, ξ, in terms of the outer variable, h, uses the definition $\xi = h/\epsilon$. Substituting this in the above equation gives

55

$$\left(\Omega_0\right)^i = K_{04}$$

$$- Cos^{-1}\left(\frac{\left\{1-\left[\dfrac{\sin(K_{03})}{1-\cos^2(K_{03})\cos^2\left[\dfrac{\delta}{\lambda}\ln\left(\tan\left(\dfrac{\pi}{4}+\dfrac{Cos^{-1}\left(\lambda Be^{-h/\epsilon}+K_{02}\right)}{2}\right)\right)+K_{05}\right]}\right]^2\right\}^{1/2}}{\cos(K_{03})}\right)$$

Since $\exp(-1/\epsilon)$ decreases to zero rapidly as $\epsilon\to 0$, $e^{-h/\epsilon}=0$. Therefore

$$\lambda Be^{-h/\epsilon}+K_{02}=K_{02}$$

Thus, the outer expansion of the ϵ^0 inner expansion solution for Ω is

$$\left[\left(\Omega_0\right)^i\right]^o = K_{04}-Cos^{-1}\left(\frac{\cos\left\{\left[\left(\alpha_0\right)^i\right]^o\right\}}{\cos(K_{03})}\right) \tag{4.74}$$

$$= K_{04}-Cos^{-1}\left(\frac{\left\{1-\left[\dfrac{\sin(K_{03})}{1-\cos^2(K_{03})\cos^2\left[\dfrac{\delta}{\lambda}\ln\left(\tan\left(\dfrac{\pi}{4}+\dfrac{Cos^{-1}(K_{02})}{2}\right)\right)+K_{05}\right]}\right]^2\right\}^{1/2}}{\cos(K_{03})}\right)$$

<u>Matching Zero Order ϵ Solutions</u>. Van Dyke's matching principle is now used to correlate the inner/outer expansions and outer/inner expansions. From Eq (4.64) the matching principle states, where Y is an arbitrary expansion

$$\left[\left(Y\right)^o\right]^i = \left[\left(Y\right)^i\right]^o$$

Thus, the inner/outer expansions and outer/inner expansions derived above are equated, resulting the the original inner expansion constants of

56

integration, K_{0i}, being expressed in terms of the original outer expansion constants of integration, C_{0i}. These outer expansion constants of integration are derived from the initial conditions of the planetary entry problem being examined.

Blending q_0 Matching Expansions. Applying the matching principle, Eq (4.64), to the inner/outer and outer/inner expansions, Eqs (4.66) and (4.71), for q_0 gives

$$K_{02} = \sqrt{\frac{C_{01}}{2 - C_{01} C_{02}}} \qquad (4.75)$$

Blending u_0 Matching Expansions. Applying the matching principle to the inner/outer and outer/inner expansions, Eqs (4.65) and (4.70), for u_0 gives

$$C_{01} = K_{01} K_{02}^{2} \exp\left[-\frac{2}{\lambda} \text{Cos}^{-1}(K_{02})\right]$$

Solving for K_{01} gives

$$K_{01} = (2 - C_{01} C_{02}) \exp\left[\frac{2}{\lambda} \text{Cos}^{-1}(K_{02})\right] \qquad (4.76)$$

Blending α_0 Matching Expansions. Applying the matching principle to the inner/outer and outer/inner expansions, Eqs (4.69) and (4.73), for α_0 gives

$$\text{Sin}^{-1}\left[\frac{1 - C_{01}}{\sqrt{1 - C_{01}^{2} C_{02}}}\right] + C_{05} = \text{Sin}^{-1}\left(\frac{\sin(K_{03})}{\sin\left\{\left[(I_0)^i\right]^0\right\}}\right)$$

Using the matching principle and Eq (4.67), $C_{03} = \left[\left(I_0\right)^o\right]^i - \left[\left(I_0\right)^i\right]^o$.

Substituting this in the above relation and solving for K_{03} gives

$$K_{03} = \operatorname{Sin}^{-1}\left\{ \sin(C_{03}) \sin\left[\operatorname{Sin}^{-1}\left(\frac{1 - C_{01}}{\sqrt{1 - C_{01}{}^2 C_{02}}}\right) + C_{05} \right] \right\} \qquad (4.77)$$

<u>Blending I_0 Matching Expansions</u>. Applying the matching principle to the inner/outer and outer/inner expansions, Eqs (4.67) and (4.72), for I_0 gives

$$C_{03} = \operatorname{Cos}^{-1}\left\{ \cos(K_{03})\cos\left[\frac{\delta}{\lambda} \ln\left(\tan\left(\frac{\pi}{4} + \frac{\operatorname{Cos}^{-1}(K_{02})}{2}\right)\right) + K_{05} \right] \right\}$$

Solving for K_{05} gives

$$K_{05} = \operatorname{Cos}^{-1}\left[\frac{\cos(C_{03})}{\cos(K_{03})}\right] - \frac{\delta}{\lambda} \ln\left(\tan\left(\frac{\pi}{4} + \frac{\operatorname{Cos}^{-1}(K_{02})}{2}\right)\right) \qquad (4.78)$$

<u>Blending Ω_0 Matching Expansions</u>. Applying the matching principle to the inner/outer and outer/inner expansions, Eqs (4.68) and (4.74), for Ω_0 gives

$$C_{04} = K_{04} - \operatorname{Cos}^{-1}\left(\frac{\cos\left\{\left[\left(\alpha_0\right)^i\right]^o\right\}}{\cos(K_{03})}\right)$$

Using the matching principle and Eq (4.69)

$$\left[\left(\alpha_0\right)^o\right]^i - \left[\left(\alpha_0\right)^i\right]^o = \operatorname{Sin}^{-1}\left[\frac{1 - C_{01}}{\sqrt{1 - C_{01}{}^2 C_{02}}}\right] + C_{05}$$

Substituting this in the above relation and solving for K_{04} gives

58

$$K_{04} = C_{04} + Cos^{-1}\left(\frac{\cos\left[Sin^{-1}\left[\frac{1 - C_{01}}{\sqrt{1 - C_{01}^2 C_{02}}}\right] + C_{05}\right]}{\cos(K_{03})}\right)$$ (4.79)

Zero Order ϵ Solutions to the Equations of Motion

The above derivations provide two separate, outer and inner, expansions; each is valid in part of the altitude domain, but neither is valid over the entire domain. Additionally, since the altitude where the inner expansion is more accurate than the outer expansion is not precisely known, where to switch from the outer, gravity dominated, expansion to the inner, air drag dominated, expansion is a not known. To generate an expansion or solution valid over the entire domain, the inner and outer expansions are blended to form a composite expansion. Using this composite expansion negates the requirement to switch from the outer to inner expansions, at an ambiguous altitude, to obtain the solution to the planetary entry problem. The composite expansion for ϵ^0 solutions is defined as (Nayfeh, 1981:277)

$$Y_0^c = Y_0^o + Y_0^i - \left[Y_0^i\right]^o = Y_0^o + Y_0^i - \left[Y_0^o\right]^i$$ (4.80)

The outer/inner (or inner/outer) expansion above accounts for the components common between the inner and outer expansions. Thus, in the altitude domain controlled by the outer expansion, the inner expansion is negated by the outer/inner expansion and vice versa. The two possibilities above are equivalent as defined by the matching principle, Eq (4.64). Throughout the composite expansions below, $\left[Y_0^o\right]^i$ is used, unless otherwise

59

noted, since it typically is a simpler and more compact expression than $\left[Y_0^i \right]^o$.

u_0 Composite Expansion. Applying the composite expansion definition, Eq (4.80), to the outer, inner and inner/outer expansions, Eqs (4.22), (4.54) and (4.65) respectively, for u_0 gives

$$u_0^c = -C_{01}\frac{h}{1+h} + K_{01}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)^2 \exp\left[-\frac{2}{\lambda}\cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)\right] \quad (4.81)$$

where K_{01} and K_{02} are given by Eqs (4.76) and (4.75).

q_0 Composite Expansion. Applying the composite expansion definition to the outer, inner and outer/inner (here this expansion is simpler than the inner/outer expansion) expansions, Eqs (4.23), (4.55) and (4.71) respectively, for q_0 gives

$$q_0^c = \frac{C_{01}\sqrt{C_{02}}}{\sqrt{1 - \left[C_{01}C_{02}(1+h) - 1\right]^2}} + \lambda Be^{-h/\epsilon} \quad (4.82)$$

I_0 Composite Expansion. Applying the composite expansion definition, to the outer, inner and inner/outer expansions, Eqs (4.24), (4.56) and (4.67) respectively, for I_0 gives

$$I_0^c = \cos^{-1}\left[\cos(K_{03})\cos\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4} + \frac{\cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)}{2}\right)\right)\right] + K_{05}\right] \quad (4.83)$$

where K_{02}, K_{03} and K_{05} are given by Eqs (4.75), (4.77) and (4.78), respectively.

$\underline{\alpha_0 \text{ Composite Expansion}}$. Applying the composite expansion definition to the outer, inner and inner/outer expansions, Eqs (4.26), (4.58) and (4.69) respectively, for α_0 gives

$$\alpha_0^c = \operatorname{Sin}^{-1}\left[\frac{1 - \dfrac{C_{01}}{1+h}}{\sqrt{1 - C_{01}{}^2 C_{02}}}\right] + \operatorname{Sin}^{-1}\left(\frac{\sin(K_{03})}{\sin\left(I_0^c\right)}\right) - \operatorname{Sin}^{-1}\left[\frac{1 - C_{01}}{\sqrt{1 - C_{01}{}^2 C_{02}}}\right] \quad (4.84)$$

$$= \operatorname{Sin}^{-1}\left[\frac{1 - \dfrac{C_{01}}{1+h}}{\sqrt{1 - C_{01}{}^2 C_{02}}}\right] - \operatorname{Sin}^{-1}\left[\frac{1 - C_{01}}{\sqrt{1 - C_{01}{}^2 C_{02}}}\right]$$

$$+ \operatorname{Sin}^{-1}\left[\frac{\sin(K_{03})}{\left[\left[1 - \cos^2(K_{03})\cos^2\left[\dfrac{\delta}{\lambda}\ln\left(\tan\left(\dfrac{\pi}{4} + \dfrac{\operatorname{Cos}^{-1}\left(\lambda B e^{-\xi} + K_{02}\right)}{2}\right)\right)\right] + K_{05}\right]\right]^{\frac{1}{2}}}\right]$$

where K_{02}, K_{03} and K_{05} are given by Eqs (4.75), (4.77) and (4.78), respectively.

$\underline{\Omega_0 \text{ Composite Expansion}}$. Applying the composite expansion definition to the outer, inner and inner/outer expansions, Eqs (4.25), (4.57) and (4.68) respectively, for Ω_0 gives

$$\Omega_0^c = K_{04} - \operatorname{Cos}^{-1}\left(\frac{\cos\left(\alpha_0^c\right)}{\cos(K_{03})}\right) = K_{04} \quad (4.85)$$

$$- \operatorname{Cos}^{-1}\left(\frac{\left\{1 - \left[\frac{\sin^2(K_{03})}{\left[1 - \cos^2(K_{03})\cos^2\left[\dfrac{\delta}{\lambda}\ln\left(\tan\left(\dfrac{\pi}{4} + \dfrac{\operatorname{Cos}^{-1}\left(\lambda B e^{-h/\epsilon} + K_{02}\right)}{2}\right)\right)\right] + K_{05}\right]}\right]^2\right\}^{\frac{1}{2}}}{\cos(K_{03})}\right)$$

where K_{02}, K_{03}, K_{04} and K_{05} are given by Eqs (4.75), (4.77), (4.79) and (4.78), respectively.

First Order ϵ Matching

In the following sections, Van Dyke's matching principle is applied to the ϵ^1 inner and outer expansion solutions. The outer expansions will be rewritten in terms of the inner variable first, and then the inner expansions will be rewritten in terms of the outer variable. Both composite expansions will be expanded for ϵ and their resulting ϵ^0 terms will be equated, resulting in the inner expansion constants of integration being expressed in terms of the outer expansion constants of integration.

Inner Expansion of the Outer Expansion Solutions. In this section, the five outer expansion solutions are rewritten in terms of the inner variable and expanded to obtain ϵ^0 terms.

Matching Expansion for u_1. From Eq (4.27), the ϵ^1 outer expansion solution for u is

$$\left(u_1\right)^o = \frac{C_{11}}{1+h}$$

Rewriting the outer variable, h, in terms of the inner variable, ξ, uses the definition $h = \epsilon \xi$. Substituting this in the above equation gives

$$\left(u_1\right)^o = \frac{C_{11}}{1+\epsilon \xi}$$

Using the binomial expansion derived as Eq (C.16) gives

$$\frac{1}{1 + \epsilon \xi} = 1 - \xi \epsilon + \xi^2 \epsilon^2 + O(\epsilon^3) = 1 + O(\epsilon)$$

Thus, the inner expansion of the ϵ^1 outer expansion solution for u is

$$\left[\left(u_1\right)^o\right]^i = C_{11} \tag{4.86}$$

Matching Expansion for q_1.. From Eq (4.28), the ϵ^1 outer expansion solution for q is

$$\left(q_1\right)^o = \frac{\left(\frac{C_{11}}{\sqrt{C_{01}(1 + h)}} + C_{12}\sqrt{1 + h}\right)}{\left(2 - C_{01}C_{02}(1 + h)\right)^{\frac{3}{2}}}$$

Rewriting the outer variable, h, in terms of the inner variable, ξ, uses the definition $h = \epsilon \xi$. Substituting this in the above equation gives

$$\left(q_1\right)^o = \frac{\left(\frac{C_{11}}{\sqrt{C_{01}(1 + \epsilon \xi)}} + C_{12}\sqrt{1 + \epsilon \xi}\right)}{\left(2 - C_{01}C_{02}(1 + \epsilon \xi)\right)^{\frac{3}{2}}}$$

Using the binomial expansion derived as Eqs (C.9) and (C.17) gives

$$\frac{1}{\sqrt{1 + \epsilon \xi}} = 1 + O(\epsilon) \text{ and } \sqrt{1 + \epsilon \xi} = 1 + O(\epsilon)$$

$$\left(2 - C_{01}C_{02}(1 + \epsilon \xi)\right)^{-\frac{3}{2}} = \left((2 - C_{01}C_{02}) - C_{01}C_{02}\epsilon \xi\right)^{-\frac{3}{2}}$$

$$= \sum_{k=0}^{\infty} \binom{-3/2}{k} (2 - C_{01}C_{02})^{-3/2 - k} (-C_{01}C_{02}\epsilon \xi)^k$$

63

$$- \sum_{k=0}^{\infty} \frac{(-1)^k \, (3/2)_k \, (2-C_{01}C_{02})^{-3/2-k}}{(k!)} \, (-C_{01}C_{02} \, \epsilon \, \xi)^k$$

$$- \left(2 - C_{01}C_{02}\right)^{-\frac{3}{2}} + O(\epsilon)$$

Thus, the inner expansion of the ϵ^1 outer expansion solution for q is

$$\left[(q_1)^o\right]^i - \frac{\left(\dfrac{C_{11}}{\sqrt{C_{01}}} + C_{12}\right)}{\left(2 - C_{01}C_{02}\right)^{\frac{3}{2}}} \tag{4.87}$$

Matching Expansion for I_1. From Eq (4.29), the ϵ^1 outer expansion solution for I is

$$\left(I_1\right)^o - C_{13}$$

By inspection, since there is no dependence on h, the inner expansion of the ϵ^1 outer expansion solution for I is

$$\left[(I_1)^o\right]^i - C_{13} \tag{4.88}$$

Matching Expansion for Ω_1. From Eq (4.30), the ϵ^1 outer expansion solution for Ω is

$$\left(\Omega_1\right)^o - C_{14}$$

By inspection, since there is no dependence on h, the inner expansion of the ϵ^1 outer expansion solution for Ω is

$$\left[(\Omega_1)^o\right]^i - C_{14} \tag{4.89}$$

Matching Expansion for α_1.. From Eq (4.31), the ϵ^1 outer expansion solution for α is

$$(\alpha_1)^o = \frac{\left(C_{12} + C_{11}C_{02}\sqrt{C_{01}}\right)(1 + h) - \left(\frac{C_{11}}{\sqrt{C_{01}}} + C_{01}C_{12}\right)}{\left(1 - C_{01}{}^2C_{02}\right)\sqrt{-C_{01} + 2(1 + h) - C_{01}C_{02}(1 + h)^2}} + C_{15}$$

Rewriting the outer variable, h, in terms of the inner variable, ξ, uses the definition $h = \epsilon \xi$. Substituting this in the above equation gives

$$(\alpha_1)^o = \frac{\left(C_{12} + C_{11}C_{02}\sqrt{C_{01}}\right)(1 + \epsilon \xi) - \left(\frac{C_{11}}{\sqrt{C_{01}}} + C_{01}C_{12}\right)}{\left(1 - C_{01}{}^2C_{02}\right)\sqrt{-C_{01} + 2(1 + \epsilon \xi) - C_{01}C_{02}(1 + \epsilon \xi)^2}} + C_{15}$$

Using the binomial expansion derived as Eq (C.17) gives

$1 + \epsilon \xi = 1 + O(\epsilon)$ and

$$\frac{1}{\left(-C_{01}C_{02}\epsilon^2 \xi^2 + 2(1 - C_{01}C_{02}) \epsilon \xi + 2 - C_{01}(C_{02} + 1)\right)^{1/2}}$$

$$= \frac{1}{\left(2 - C_{01}(C_{02} + 1)\right)^{1/2}} + O(\epsilon)$$

Thus, the inner expansion of the ϵ^1 outer expansion solution for α is

$$\left[(\alpha_1)^o\right]^i = \frac{C_{12}\left(C_{01} + 1\right) + \frac{C_{11}}{\sqrt{C_{01}}}\left(C_{01}C_{02} - 1\right)}{\left(1 - C_{01}{}^2C_{02}\right)\sqrt{2 - C_{01}(C_{02} + 1)}} + C_{15} \qquad (4.90)$$

65

<u>Outer Expansion of the Inner Expansion Solutions.</u> In this section, the five inner expansion solutions are rewritten in terms of the outer variable and expanded to obtain ϵ^0 terms.

<u>Matching Expansion for u_1.</u> From Eq (4.59), the ϵ^1 inner expansion solution for u is

$$
(u_1)^i = \left(\lambda Be^{-\xi} + K_{02}\right)^2 \exp\left(-\frac{2}{\lambda}\cos^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)\right)
$$

$$
\times \left[K_{11} + \frac{2K_{01}\left(K_{12} + K_{02}K_{121}\right)(\xi + 1)}{\lambda Be^{-\xi} + K_{02}} - \frac{2K_{121}K_{01}}{\lambda}\sin^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)\right.
$$

$$
+ \frac{\frac{2}{\lambda}\left[K_{01}K_{12} - K_{01}K_{121}\right]\left(K_{02}\xi - \lambda Be^{-\xi}\right)}{\sqrt{1 - \left(\lambda Be^{-\xi} + K_{02}\right)^2}} + \frac{2K_{02}K_{01}K_{121}}{\lambda\sqrt{1 - K_{02}^2}}
$$

$$
\left.\times \ln\left[\frac{2\sqrt{(1-K_{02}^2)\left[1 - (\lambda Be^{-\xi} + K_{02})^2\right]} - 2K_{02}\left(\lambda Be^{-\xi} + K_{02}\right) + 2}{\lambda Be^{-\xi}}\right]\right\}
$$

Rewriting the inner variable, ξ, in terms of the outer variable, h, uses the definition $\xi = h/\epsilon$. Substituting this in the above equation gives

$$
(u_1)^i = \left(\lambda Be^{-h/\epsilon} + K_{02}\right)^2 \exp\left(-\frac{2}{\lambda}\cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)\right)
$$

$$
\times \left[K_{11} + \frac{2K_{01}\left(K_{12} + K_{02}K_{121}\right)(\xi + 1)}{\lambda Be^{-h/\epsilon} + K_{02}} - \frac{2K_{121}K_{01}}{\lambda}\sin^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)\right.
$$

$$
+ \frac{\frac{2}{\lambda}\left[K_{01}K_{12} - K_{01}K_{121}\right]\left(K_{02}\xi - \lambda Be^{-h/\epsilon}\right)}{\sqrt{1 - \left(\lambda Be^{-h/\epsilon} + K_{02}\right)^2}} + \frac{2K_{02}K_{01}K_{121}}{\lambda\sqrt{1 - K_{02}^2}}\left[\xi\right.
$$

66

$$+ \ln\left(\frac{2\sqrt{(1-K_{02}{}^2)\left[1-(\lambda Be^{-h/\epsilon}+K_{02})^2\right]} - 2K_{02}\left(\lambda Be^{-h/\epsilon} + K_{02}\right) + 2}{\lambda B}\right)\Biggr]\Biggr]\Biggr\}$$

Since $\exp(-1/\epsilon)$ decreases to zero rapidly as $\epsilon \to 0$, $e^{-h/\epsilon} = 0$ (Nayfeh, 1981:260). Therefore

$$\lambda Be^{-h/\epsilon} + K_{02} = K_{02}$$

Thus, the outer expansion of the ϵ^1 inner expansion solution for u is

$$\left[(u_1)^i\right]^o = K_{02}{}^2 \exp\left(-\frac{2}{\lambda}\cos^{-1}(K_{02})\right)\left\{K_{11} + 2K_{01}\left(\frac{K_{12}}{K_{02}} + K_{121}\right)(\xi + 1)\right.$$

$$\left. - \frac{2K_{121}K_{01}}{\lambda}\sin^{-1}(K_{02}) + \frac{2K_{01}K_{02}}{\lambda\sqrt{1-K_{02}{}^2}}\left[K_{12}\xi + K_{121}\ln\left(\frac{4(1-K_{02}{}^2)}{\lambda B}\right)\right]\right\} \quad (4.91)$$

Matching Expansion for q_1. From Eq (4.60), the ϵ^1 inner expansion solution for q is

$$\left(q_1\right)^i = K_{121}\left(K_{02}\xi - \lambda Be^{-\xi}\right) + K_{12}$$

Rewriting the inner variable, ξ, in terms of the outer variable, h, uses the definition $\xi = h/\epsilon$. Substituting this in the above equation gives

$$\left(q_1\right)^i = K_{121}\left(K_{02}\xi - \lambda Be^{-h/\epsilon}\right) + K_{12}$$

Since $\exp(-1/\epsilon)$ decreases to zero rapidly as $\epsilon \to 0$, $e^{-h/\epsilon} = 0$ (Nayfeh, 1981:260). Therefore

$$\lambda Be^{-h/\epsilon} = 0$$

Thus, the outer expansion of the ϵ^1 inner expansion solution for q is

67

$$\left[\left(q_1\right)^i\right]^o = K_{121}K_{02}\,\xi + K_{12} \tag{4.92}$$

Matching Expansion for I_1. From Eq (4.61), the ϵ^1 inner expansion solution for I is

$$\left(I_1\right)^i = K_{13} + \frac{\delta}{\lambda}\left\{\frac{K_{131}\left(K_{132} - \frac{K_{133}}{\lambda Be^{-\xi} + K_{02}}\right)}{\sqrt{1 - \left(\lambda Be^{-\xi} + K_{02}\right)^2}}\right.$$

$$+ \left[K_{131}K_{132} + K_{134}\left(\frac{\pi}{2} - K_{15}\right)\right]\ln\left[\tan\left(\frac{\pi}{4} + \frac{\mathrm{Cos}^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)}{2}\right)\right]$$

$$+ K_{134}\left[\frac{K_{02}\,\mathrm{Cos}^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)}{\sqrt{1 - K_{02}^2}} - \frac{\left[\mathrm{Cos}^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)\right]^2}{2}\right.$$

$$\left.\left.- \frac{\left[\mathrm{Cos}^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)\right]^4}{8}\right]\right\}$$

Rewriting the inner variable, ξ, in terms of the outer variable, h, uses the definition $\xi = h/\epsilon$. Substituting this in the above equation gives

$$\left(I_1\right)^i = K_{13} + \frac{\delta}{\lambda}\left\{\frac{K_{131}\left(K_{132} - \frac{K_{133}}{\lambda Be^{-h/\epsilon} + K_{02}}\right)}{\sqrt{1 - \left(\lambda Be^{-h/\epsilon} + K_{02}\right)^2}}\right.$$

$$+ \left[K_{131}K_{132} + K_{134}\left(\frac{\pi}{2} - K_{15}\right)\right]\ln\left[\tan\left(\frac{\pi}{4} + \frac{\mathrm{Cos}^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)}{2}\right)\right]$$

68

$$+ K_{134} \left[\frac{K_{02} \cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02} \right)}{\sqrt{1 - K_{02}^2}} - \frac{\left[\cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02} \right) \right]^2}{2} \right.$$

$$\left. \left. - \frac{\left[\cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02} \right) \right]^4}{8} \right] \right\}$$

Since $\exp(-1/\epsilon)$ decreases to zero rapidly as $\epsilon \to 0$, $e^{-h/\epsilon} = 0$ (Nayfeh, 1981:260). Therefore

$$\lambda Be^{-h/\epsilon} + K_{02} = K_{02}$$

Thus, the outer expansion of the ϵ^1 inner expansion solution for I is

$$\left[\left(I_1 \right)^i \right]^0 = K_{13} + \frac{\delta}{\lambda} \left\{ \frac{K_{131}\left(K_{132} - \frac{K_{133}}{K_{02}} \right)}{\sqrt{1 - K_{02}^2}} \right.$$

$$+ \left[K_{131}K_{132} + K_{134}\left(\frac{\pi}{2} - K_{15} \right) \right] \ln\left[\tan\left(\frac{\pi}{4} + \frac{\cos^{-1}(K_{02})}{2} \right) \right]$$

$$+ K_{134}\left[\frac{K_{02}\cos^{-1}(K_{02})}{\sqrt{1 - K_{02}^2}} - \frac{\left[\cos^{-1}(K_{02}) \right]^2}{2} - \frac{\left[\cos^{-1}(K_{02}) \right]^4}{8} \right] \right\} \quad (4.93)$$

<u>Matching Expansion for α_1.</u> From Eq (4.63), the ϵ^1 inner expansion solution for α is

$$\left(\alpha_1 \right)^i = K_{15} - \sin^{-1}\left(\lambda Be^{-\xi} + K_{02} \right) + \frac{K_{02}}{\sqrt{1 - K_{02}^2}} \left\{ \xi \right.$$

$$\left. + \ln\left[\frac{2\sqrt{\left(1 - K_{02}^2 \right)\left[1 - \left(\lambda Be^{-\xi} + K_{02} \right)^2 \right]} - 2K_{02}\left(\lambda Be^{-\xi} + K_{02} \right) + 2}{\lambda B} \right] \right\}$$

69

Rewriting the inner variable, ξ, in terms of the outer variable, h, uses the definition $\xi = h/\epsilon$. Substituting this in the above equation gives

$$\left(\alpha_1\right)^i = K_{15} - \text{Sin}^{-1}\left(\lambda B e^{-h/\epsilon} + K_{02}\right) + \frac{K_{02}}{\sqrt{1-K_{02}^2}}\left\{\xi\right.$$

$$\left. + \ln\left[\frac{2\sqrt{(1-K_{02}^2)\left[1-\left(\lambda B e^{-h/\epsilon}+K_{02}\right)^2\right]} - 2K_{02}\left(\lambda B e^{-h/\epsilon}+K_{02}\right) + 2}{\lambda B}\right]\right\}$$

Since $\exp(-1/\epsilon)$ decreases to zero rapidly as $\epsilon \to 0$, $e^{-h/\epsilon} = 0$ (Nayfeh, 1981:260). Therefore

$$\lambda B e^{-h/\epsilon} + K_{02} = K_{02}$$

Thus, the outer expansion of the ϵ^1 inner expansion solution for α is

$$\left[\left(\alpha_1\right)^i\right]^0 = K_{15} - \text{Sin}^{-1}(K_{02}) + \frac{K_{02}}{\sqrt{1-K_{02}^2}}\left\{\xi + \ln\left[\frac{4\left(1-K_{02}^2\right)}{\lambda B}\right]\right\} \qquad (4.94)$$

<u>Matching Expansion for Ω_1</u>. From Eq (4.62), the ϵ^1 inner expansion solution for Ω is

$$\left(\Omega_1\right)^i = K_{14} + \frac{\delta}{\lambda}\left\{\frac{K_{141}\left(K_{132} - \frac{K_{133}}{\lambda B e^{-\xi} + K_{02}}\right)}{\sqrt{1-\left(\lambda B e^{-\xi}+K_{02}\right)^2}} + \frac{K_{142}K_{133}}{\lambda B e^{-\xi}+K_{02}}\right.$$

$$+ K_{143}\ln\left[\tan\left(\frac{\pi}{4} + \frac{\text{Cos}^{-1}\left(\lambda B e^{-\xi}+K_{02}\right)}{2}\right)\right]$$

$$+ \left(K_{132}K_{142} + K_{144}\left(\frac{\pi}{2}-K_{15}\right)\right)\left[\frac{4\ln\left(\lambda B e^{-\xi}+K_{02}\right)}{3} - \frac{\left(\lambda B e^{-\xi}+K_{02}\right)^2}{6}\right]$$

70

$$+ \frac{K_{02} K_{144} \cos^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)}{\sqrt{1 - K_{02}{}^2}}$$

$$+ K_{145} \left[\frac{\left[\cos^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)\right]^3}{6} + \frac{3\left[\cos^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)\right]^5}{40} \right]$$

$$+ K_{146} \left[\frac{\left[\cos^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)\right]^2}{2} + \frac{\left[\cos^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)\right]^4}{8} \right] \Bigg\}$$

Rewriting the inner variable, ξ, in terms of the outer variable, h, uses the definition $\xi = h/\epsilon$. Substituting this in the above equation gives

$$\left(\Omega_1\right)^i = K_{14} + \frac{\delta}{\lambda} \left\{ \frac{K_{141}\left(K_{132} - \dfrac{K_{133}}{\lambda Be^{-h/\epsilon} + K_{02}}\right)}{\sqrt{1 - \left(\lambda Be^{-h/\epsilon} + K_{02}\right)^2}} + \frac{K_{142} K_{133}}{\lambda Be^{-h/\epsilon} + K_{02}} \right.$$

$$+ K_{143} \ln\left[\tan\left(\frac{\pi}{4} + \frac{\cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)}{2}\right)\right]$$

$$+ \left(K_{132}K_{142} + K_{144}\left(\frac{\pi}{2} - K_{15}\right)\right)\left[\frac{4 \ln\left(\lambda Be^{-h/\epsilon} + K_{02}\right)}{3} - \frac{\left(\lambda Be^{-h/\epsilon} + K_{02}\right)^2}{6}\right]$$

$$+ \frac{K_{02} K_{144} \cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)}{\sqrt{1 - K_{02}{}^2}}$$

$$+ K_{145} \left[\frac{\left[\cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)\right]^3}{6} + \frac{3\left[\cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)\right]^5}{40} \right]$$

$$+ K_{146} \left[\frac{\left[\cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)\right]^2}{2} + \frac{\left[\cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)\right]^4}{8} \right] \Bigg\}$$

71

Since $\exp(-1/\epsilon)$ decreases to zero rapidly as $\epsilon \to 0$, $e^{-h/\epsilon} = 0$ (Nayfeh, 1981:260). Therefore

$$\lambda \, Be^{-h/\epsilon} + K_{02} = K_{02}$$

Thus, the outer expansion of the ϵ^1 inner expansion solution for Ω is

$$
\left[(\Omega_i)^i \right]^o = K_{14} + \frac{\delta}{\lambda} \left\{ \frac{K_{141}\left(K_{132} - \dfrac{K_{133}}{K_{02}}\right)}{\sqrt{1 - K_{02}^2}} + \frac{K_{142} K_{133}}{K_{02}} \right.
$$

$$
+ K_{143} \ln\left[\tan\left(\frac{\pi}{4} + \frac{\mathrm{Cos}^{-1}(K_{02})}{2}\right) \right]
$$

$$
+ \left(K_{132}K_{142} + K_{144}\left(\frac{\pi}{2} - K_{15}\right)\right)\left[\frac{4 \ln(K_{02})}{3} - \frac{K_{02}^2}{6}\right]
$$

$$
+ \frac{K_{02} K_{144} \mathrm{Cos}^{-1}(K_{02})}{\sqrt{1 - K_{02}^2}}
$$

$$
+ K_{145}\left[\frac{\left[\mathrm{Cos}^{-1}(K_{02})\right]^3}{6} + \frac{3\left[\mathrm{Cos}^{-1}(K_{02})\right]^5}{40}\right]
$$

$$
\left. + K_{146}\left[\frac{\left[\mathrm{Cos}^{-1}(K_{02})\right]^2}{2} + \frac{\left[\mathrm{Cos}^{-1}(K_{02})\right]^4}{8}\right] \right\}
\tag{4.95}
$$

Matching First Order ϵ Solutions. Van Dyke's matching principle is now used to correlate the inner/outer expansions and outer/inner expansions. From Eq (4.64) the matching principle states, where Y is an arbitrary expansion

$$\left[(Y)^o\right]^i = \left[(Y)^i\right]^o$$

Thus, the inner/outer expansions and outer/inner expansions derived above are equated, resulting in the original inner expansion constants of integration, K_{1i}, being expressed in terms of the original outer expansion constants of integration, C_{1i}. These outer expansion constants of integration are derived from the initial conditions of the planetary entry problem being examined.

Blending q_1 Matching Expansions. Applying the matching principle, Eq (4.64), to the inner/outer and outer/inner expansions, Eqs (4.87) and (4.92), for q_1 gives

$$\frac{\left(\frac{C_{11}}{\sqrt{C_{01}}} + C_{12}\right)}{\left(2 - C_{01}C_{02}\right)^{\frac{3}{2}}} - K_{121}K_{02}\,\xi + K_{12}$$

Solving for K_{12} gives

$$K_{12} - \frac{\left(\frac{C_{11}}{\sqrt{C_{01}}} + C_{12}\right)}{\left(2 - C_{01}C_{02}\right)^{\frac{3}{2}}} - K_{121}K_{02}\,\xi \qquad (4.96)$$

Blending u_1 Matching Expansions. Applying the matching principle to the inner/outer and outer/inner expansions, Eqs (4.86) and (4.91), for u_1 gives

$$C_{11} - K_{02}{}^2 \exp\left(-\frac{2}{\lambda}\cos^{-1}(K_{02})\right)\left\{\left[K_{11} + 2K_{01}\left(\frac{K_{12}}{K_{02}} + K_{121}\right)(\xi + 1)\right.\right.$$

$$\left.\left. - \frac{2K_{121}K_{01}}{\lambda}\sin^{-1}(K_{02}) + \frac{2K_{01}K_{02}}{\lambda\sqrt{1 - K_{02}{}^2}}\left[K_{12}\,\xi + K_{121}\,\ln\left(\frac{4(1-K_{02}{}^2)}{\lambda B}\right)\right]\right]\right\}$$

73

Solving for K_{11} gives

$$K_{11} = \frac{K_{01} C_{11}}{C_{01}} - \left[2K_{01} \left(\frac{K_{12}}{K_{02}} + K_{121} \right) (\xi + 1) \right] + \frac{2K_{121} K_{01}}{\lambda} \, \text{Sin}^{-1}(K_{02})$$

$$- \frac{2K_{01} K_{02}}{\lambda \sqrt{1 - K_{02}^2}} \left[K_{12} \xi + K_{121} \ln \left(\frac{4(1 - K_{02}^2)}{\lambda B} \right) \right] \qquad (4.97)$$

Blending α_1 Matching Expansions. Applying the matching principle to the inner/outer and outer/inner expansions, Eqs (4.90) and (4.94), for α_1 gives

$$\frac{C_{12} (C_{01} + 1) + \frac{C_{11}}{\sqrt{C_{01}}} (C_{01} C_{02} - 1)}{\left(1 - C_{01}^2 C_{02} \right) \sqrt{2 - C_{01}(C_{02} + 1)}} + C_{15} = K_{15} - \text{Sin}^{-1}(K_{02})$$

$$+ \frac{K_{02}}{\sqrt{1 - K_{02}^2}} \left\{ \xi + \ln \left[\frac{4 \left(1 - K_{02}^2 \right)}{\lambda B} \right] \right\}$$

Solving for K_{15} gives

$$K_{15} = C_{15} + \frac{C_{12} (C_{01} + 1) + \frac{C_{11}}{\sqrt{C_{01}}} (C_{01} C_{02} - 1)}{\left(1 - C_{01}^2 C_{02} \right) \sqrt{2 - C_{01}(C_{02} + 1)}} + \text{Sin}^{-1}(K_{02})$$

$$- \frac{K_{02}}{\sqrt{1 - K_{02}^2}} \left\{ \xi + \ln \left[\frac{4 \left(1 - K_{02}^2 \right)}{\lambda B} \right] \right\} \qquad (4.98)$$

Blending I_1 Matching Expansions. Applying the matching principle to the inner/outer and outer/inner expansions, Eqs (4.88) and (4.93), for I_1 gives

74

$$C_{13} = K_{13} + \frac{\delta}{\lambda} \left\{ \frac{K_{131}\left(K_{132} - \frac{K_{133}}{K_{02}}\right)}{\sqrt{1 - K_{02}^2}} \right.$$

$$+ \left[K_{131}K_{132} + K_{134}\left(\frac{\pi}{2} - K_{15}\right) \right] \ln\left[\tan\left(\frac{\pi}{4} + \frac{\cos^{-1}(K_{02})}{2}\right) \right]$$

$$\left. + K_{134}\left[\frac{K_{02}\cos^{-1}(K_{02})}{\sqrt{1 - K_{02}^2}} - \frac{\left[\cos^{-1}(K_{02})\right]^2}{2} - \frac{\left[\cos^{-1}(K_{02})\right]^4}{8} \right] \right\}$$

Solving for K_{13} gives

$$K_{13} = C_{13} - \frac{\delta}{\lambda} \left\{ \frac{K_{131}\left(K_{132} - \frac{K_{133}}{K_{02}}\right)}{\sqrt{1 - K_{02}^2}} \right.$$

$$+ \left[K_{131}K_{132} + K_{134}\left(\frac{\pi}{2} - K_{15}\right) \right] \ln\left[\tan\left(\frac{\pi}{4} + \frac{\cos^{-1}(K_{02})}{2}\right) \right]$$

$$\left. + K_{134}\left[\frac{K_{02}\cos^{-1}(K_{02})}{\sqrt{1 - K_{02}^2}} - \frac{\left[\cos^{-1}(K_{02})\right]^2}{2} - \frac{\left[\cos^{-1}(K_{02})\right]^4}{8} \right] \right\} \qquad (4.99)$$

Blending Ω_1 Matching Expansions. Applying the matching principle to the inner/outer and outer/inner expansions, Eqs (4.89) and (4.95), for Ω_1 gives

$$C_{14} = K_{14} + \frac{\delta}{\lambda} \left\{ \frac{K_{141}\left(K_{132} - \frac{K_{133}}{K_{02}}\right)}{\sqrt{1 - K_{02}^2}} + \frac{K_{142}\,K_{133}}{K_{02}} \right.$$

$$+ K_{143} \ln\left[\tan\left(\frac{\pi}{4} + \frac{\cos^{-1}(K_{02})}{2}\right) \right]$$

$$+ \left(K_{132}K_{142} + K_{144}\left(\frac{\pi}{2} - K_{15}\right)\right)\left[\frac{4\ln(K_{02})}{3} - \frac{K_{02}^2}{6}\right] + \frac{K_{02}K_{144}\cos^{-1}(K_{02})}{\sqrt{1 - K_{02}^2}}$$

$$+ K_{145}\left[\frac{\left[\cos^{-1}(K_{02})\right]^3}{6} + \frac{3\left[\cos^{-1}(K_{02})\right]^5}{40}\right]$$

$$+ K_{146}\left[\frac{\left[\cos^{-1}(K_{02})\right]^2}{2} + \frac{\left[\cos^{-1}(K_{02})\right]^4}{8}\right]\Bigg\}$$

Solving for K_{14} gives

$$K_{14} = C_{14} - \frac{\delta}{\lambda}\Bigg\{\frac{K_{141}\left(K_{132} - \frac{K_{133}}{K_{02}}\right)}{\sqrt{1 - K_{02}^2}} + \frac{K_{142}K_{133}}{K_{02}}$$

$$+ K_{143}\ln\left[\tan\left(\frac{\pi}{4} + \frac{\cos^{-1}(K_{02})}{2}\right)\right]$$

$$+ \left(K_{132}K_{142} + K_{144}\left(\frac{\pi}{2} - K_{15}\right)\right)\left[\frac{4\ln(K_{02})}{3} - \frac{K_{02}^2}{6}\right] + \frac{K_{02}K_{144}\cos^{-1}(K_{02})}{\sqrt{1 - K_{02}^2}}$$

$$+ K_{145}\left[\frac{\left[\cos^{-1}(K_{02})\right]^3}{6} + \frac{3\left[\cos^{-1}(K_{02})\right]^5}{40}\right]$$

$$+ K_{146}\left[\frac{\left[\cos^{-1}(K_{02})\right]^2}{2} + \frac{\left[\cos^{-1}(K_{02})\right]^4}{8}\right]\Bigg\} \tag{4.100}$$

First Order ϵ Solutions to the Equations of Motion

As in the derivation of ϵ^0 solutions to the equations of motion, the composite expansion or solution is used to blend the distinct, but

overlapping, outer and inner expansions to give an expansion valid over the entire altitude domain. The composite expansion for ϵ^1 solutions is defined as (Nayfeh, 1981:277)

$$Y_1^c = Y_1^o + Y_1^i - \left[Y_1^i\right]^o = Y_1^o + Y_1^i - \left[Y_1^o\right]^i \qquad (4.101)$$

Also, as in the ϵ^0 derivation, $\left[Y_1^o\right]^i$ is used, unless otherwise noted, since it typically is a simpler and more compact expression than $\left[Y_1^i\right]^o$.

$\underline{u_1\ \text{Composite Expansion}}$. Applying the composite expansion definition, Eq (4.80), to the outer, inner and inner/outer expansions, Eqs (4.27), (4.59) and (4.86) respectively, for u_1 gives

$$u_1^c = -C_{11}\frac{h}{1+h} + \left(\lambda Be^{-h/\epsilon} + K_{02}\right)^2 \exp\left(-\frac{2}{\lambda}\cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)\right)$$

$$\times \left\{ K_{11} + \frac{2K_{01}\left(K_{12} + K_{02}K_{121}\right)\left(\frac{h}{\epsilon}+1\right)}{\lambda Be^{-h/\epsilon}+K_{02}} - \frac{2K_{121}K_{01}}{\lambda}\sin^{-1}\left(\lambda Be^{-h/\epsilon}+K_{02}\right) \right.$$

$$+ \frac{\frac{2}{\lambda}\left[K_{01}K_{12} - K_{01}K_{121}\right]\left(K_{02}\frac{h}{\epsilon} - \lambda Be^{-h/\epsilon}\right)}{\sqrt{1 - \left(\lambda Be^{-h/\epsilon} + K_{02}\right)^2}} + \frac{2K_{02}K_{01}K_{121}}{\lambda\sqrt{1 - K_{02}^2}}$$

$$\left. \times \ln\left[\frac{2\sqrt{(1-K_{02}^2)\left[1-(\lambda Be^{-h/\epsilon}+K_{02})^2\right]}-2K_{02}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)+2}{\lambda Be^{-h/\epsilon}}\right] \right\} \qquad (4.102)$$

where K_{01}, K_{02}, K_{12} and K_{121} are given by Eqs (4.76), (4.75), (4.96) and (E.26), respectively.

q_1 Composite Expansion. Applying the composite expansion definition to the outer, inner and outer/inner (here this expansion is simpler than the inner/outer expansion) expansions, Eqs (4.28), (4.60) and (4.92) respectively, for q_1 gives

$$q_1^c = \frac{\left(\dfrac{C_{11}}{\sqrt{C_{01}(1+h)}} + C_{12}\sqrt{1+h}\right)}{\left(2 - C_{01}C_{02}(1+h)\right)^{\frac{3}{2}}} - \lambda B e^{-h/\epsilon} \tag{4.103}$$

I_1 Composite Expansion. Applying the composite expansion definition, to the outer, inner and inner/outer expansions, Eqs (4.29), (4.61) and (4.88) respectively, for I_1 gives

$$I_1^c = K_{13} + \frac{\delta}{\lambda} \left\{ \frac{K_{131}\left(K_{132} - \dfrac{K_{133}}{\lambda B e^{-h/\epsilon} + K_{02}}\right)}{\sqrt{1 - \left(\lambda B e^{-h/\epsilon} + K_{02}\right)^2}} \right.$$

$$+ \left[K_{131}K_{132} \cdot K_{134}\left(\frac{\pi}{2} - K_{15}\right)\right] \ln\left[\tan\left(\frac{\pi}{4} + \frac{\text{Cos}^{-1}\left(\lambda B e^{-h/\epsilon} + K_{02}\right)}{2}\right)\right]$$

$$+ K_{134}\left[\frac{K_{02}\,\text{Cos}^{-1}\left(\lambda B e^{-h/\epsilon} + K_{02}\right)}{\sqrt{1 - K_{02}^2}} - \frac{\left[\text{Cos}^{-1}\left(\lambda B e^{-h/\epsilon} + K_{02}\right)\right]^2}{2}\right.$$

$$\left.\left. - \frac{\left[\text{Cos}^{-1}\left(\lambda B e^{-h/\epsilon} + K_{02}\right)\right]^4}{8}\right]\right\} \tag{4.104}$$

where K_{02}, K_{15}, K_{131}, K_{132}, K_{133}, and K_{134} are given by Eqs (4.75), (4.98) and (E.30), respectively.

$\underline{\alpha_1 \text{ Composite Expansion}}$. Applying the composite expansion definition to the outer, inner and inner/outer expansions, Eqs (4.31), (4.63) and (4.90) respectively, for α_1 gives

$$\alpha_1^c = \frac{\left(C_{12} + C_{11}C_{02}\sqrt{C_{01}}\right)(1+h) - \left(\dfrac{C_{11}}{\sqrt{C_{01}}} + C_{01}C_{12}\right)}{\left(1 - C_{01}{}^2 C_{02}\right)\sqrt{-C_{01} + 2(1+h) - C_{01}C_{02}(1+h)^2}} + K_{15}$$

$$- \text{Sin}^{-1}\left(\lambda B e^{-h/\epsilon} + K_{02}\right) + \frac{K_{02}}{\sqrt{1-K_{02}{}^2}}$$

$$\times \ln\left[\frac{2\sqrt{(1-K_{02}{}^2)\left[1 - \left(\lambda B e^{-h/\epsilon} + K_{02}\right)^2\right]} - 2K_{02}\left(\lambda B e^{-h/\epsilon} + K_{02}\right) + 2}{\lambda B e^{-h/\epsilon}}\right]$$

$$- \frac{C_{12}\left(C_{01} + 1\right) + \dfrac{C_{11}}{\sqrt{C_{01}}}\left(C_{01}C_{02} - 1\right)}{\left(1 - C_{01}{}^2 C_{02}\right)\sqrt{2 - C_{01}(C_{02} + 1)}} \tag{4.105}$$

where K_{02} and K_{15} are given by Eqs (4.75) and (4.98).

$\underline{\Omega_1 \text{ Composite Expansion}}$. Applying the composite expansion definition to the outer, inner and inner/outer expansions, Eqs (4.30), (4.62) and (4.89) respectively, for Ω_1 gives

$$\Omega_1^c = K_{14} + \frac{\delta}{\lambda}\left\{\frac{K_{141}\left(K_{132} - \dfrac{K_{133}}{\lambda B e^{-h/\epsilon} + K_{02}}\right)}{\sqrt{1 - \left(\lambda B e^{-h/\epsilon} + K_{02}\right)^2}} + \frac{K_{142}K_{133}}{\lambda B e^{-h/\epsilon} + K_{02}}\right.$$

$$+ K_{143}\ln\left[\tan\left(\frac{\pi}{4} + \frac{\cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)}{2}\right)\right]$$

$$+ \left(K_{132}K_{142} + K_{144}\left(\frac{\pi}{2} - K_{15}\right)\right)\left[\frac{4\ln\left(\lambda Be^{-h/\epsilon} + K_{02}\right)}{3} - \frac{\left(\lambda Be^{-h/\epsilon} + K_{02}\right)^2}{6}\right]$$

$$+ \frac{K_{02}K_{144}\cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)}{\sqrt{1 - K_{02}^2}}$$

$$+ K_{145}\left[\frac{\left[\cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)\right]^3}{6} + \frac{3\left[\cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)\right]^5}{40}\right]$$

$$+ K_{146}\left[\frac{\left[\cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)\right]^2}{2} + \frac{\left[\cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)\right]^4}{8}\right]\right\} \quad (4.106)$$

where K_{02}, K_{141}, K_{142}, K_{143}, K_{144}, K_{145}, and K_{146} are given by Eqs (4.75) and (E.32).

Equations of Motion

Once the composite expansions for the five dependent variables (u, q, I, Ω and α) are determined, the approximate, analytical solution to the five equations of motion are given as

$$u^c = u_0^c + \left(u_1^c\right)\epsilon + O(\epsilon^2)$$

$$= \left\{-C_{01}\frac{h}{1+h} + K_{01}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)^2 \exp\left[-\frac{2}{\lambda}\cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)\right]\right\}$$

80

$$+ \left\{ -C_{11}\frac{h}{1+h} + \left(\lambda Be^{-h/\epsilon} + K_{02}\right)^2 \exp\left(-\frac{2}{\lambda} \cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)\right) \right.$$

$$\times \left\{ K_{11} + \frac{2K_{01}\left(K_{12} + K_{02}K_{121}\right)\left(\frac{h}{\epsilon}+1\right)}{\lambda Be^{-h/\epsilon} + K_{02}} - \frac{2K_{121}K_{01}}{\lambda} \sin^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right) \right.$$

$$+ \frac{\frac{2}{\lambda}\left[K_{01}K_{12} - K_{01}K_{121}\right]\left(K_{02}\frac{h}{\epsilon} - \lambda Be^{-h/\epsilon}\right)}{\sqrt{1 - \left(\lambda Be^{-h/\epsilon} + K_{02}\right)^2}} + \frac{2K_{02}K_{01}K_{121}}{\lambda\sqrt{1-K_{02}^2}}$$

$$\left.\left.\times \ln\left[\frac{2\sqrt{(1-K_{02}^2)\left[1-(\lambda Be^{-h/\epsilon}+K_{02})^2\right]} - 2K_{02}\left(\lambda Be^{-h/\epsilon}+K_{02}\right)+2}{\lambda Be^{-h/\epsilon}}\right]\right\}\right\}\epsilon$$

$$+ O(\epsilon^2) \tag{4.107}$$

$$q^c = q_0^c + \left(q_1^c\right)\epsilon + O(\epsilon^2)$$

$$= \left\{\frac{C_{01}\sqrt{C_{02}}}{\sqrt{1 - \left[C_{01}C_{02}(1+h) - 1\right]^2}} + \lambda Be^{-h/\epsilon}\right\}$$

$$\left[\frac{\left(\frac{C_{11}}{\sqrt{C_{01}(1+h)}} + C_{12}\sqrt{1+h}\right)}{\left(2 - C_{01}C_{02}(1+h)\right)^{\frac{3}{2}}} - \lambda Be^{-h/\epsilon}\right]\epsilon + O(\epsilon^2) \tag{4.108}$$

$$I^c = I_0^c + \left(I_1^c\right)\epsilon + O(\epsilon^2)$$

$$= \cos^{-1}\left\{\cos(K_{03})\cos\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4} + \frac{\cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)}{2}\right)\right)\right] + K_{05}\right\}$$

81

$$+ \left\{ K_{13} + \frac{\delta}{\lambda} \left\{ \frac{K_{131}\left(K_{132} - \dfrac{K_{133}}{\lambda Be^{-h/\epsilon} + K_{02}}\right)}{\sqrt{1 - \left(\lambda Be^{-h/\epsilon} + K_{02}\right)^2}} \right.\right.$$

$$+ \left[K_{131}K_{132} + K_{134}\left(\frac{\pi}{2} - K_{15}\right)\right] \ln\left[\tan\left(\frac{\pi}{4} + \frac{Cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)}{2}\right)\right]$$

$$+ K_{134}\left[\frac{K_{02} Cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)}{\sqrt{1 - K_{02}^2}} - \frac{\left[Cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)\right]^2}{2}\right.$$

$$\left.\left.\left.- \frac{\left[Cos^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right)\right]^4}{8}\right]\right\}\right\} \epsilon + O(\epsilon^2) \tag{4.109}$$

$$\alpha^c = \alpha_0^c + \left(\alpha_1^c\right)\epsilon + O(\epsilon^2)$$

$$-\left\{Sin^{-1}\left[\frac{1 - \dfrac{C_{01}}{1+h}}{\sqrt{1 - C_{01}^2 C_{02}}}\right] + Sin^{-1}\left(\frac{\sin(K_{03})}{\sin(I_0^c)}\right) - Sin^{-1}\left[\frac{1 - C_{01}}{\sqrt{1 - C_{01}^2 C_{02}}}\right]\right\}$$

$$+ \left\{\frac{\left(C_{12} + C_{11}C_{02}\sqrt{C_{01}}\right)(1+h) - \left(\dfrac{C_{11}}{\sqrt{C_{01}}} + C_{01}C_{12}\right)}{\left(1 - C_{01}^2 C_{02}\right)\sqrt{-C_{01} + 2(1+h) - C_{01}C_{02}(1+h)^2}} + K_{15}\right.$$

$$- Sin^{-1}\left(\lambda Be^{-h/\epsilon} + K_{02}\right) + \frac{K_{02}}{\sqrt{1 - K_{02}^2}}$$

$$\times \ln\left[\frac{2\sqrt{(1 - K_{02}^2)\left[1 - \left(\lambda Be^{-h/\epsilon} + K_{02}\right)^2\right]} - 2K_{02}\left(\lambda Be^{-h/\epsilon} + K_{02}\right) + 2}{\lambda Be^{-h/\epsilon}}\right]$$

82

$$-\frac{C_{12}\left(C_{01}+1\right)+\frac{C_{11}}{\sqrt{C_{01}}}\left(C_{01}C_{02}-1\right)}{\left(1-C_{01}{}^2 C_{02}\right)\sqrt{2-C_{01}(C_{02}+1)}}\Bigg]\epsilon + O(\epsilon^2) \tag{4.110}$$

$$\Omega^c = \Omega_0^c + \left(\Omega_1^c\right)\epsilon + O(\epsilon^2)$$

$$= \left[K_{04} - \cos^{-1}\!\left(\frac{\cos\!\left(\alpha_0^c\right)}{\cos(K_{03})}\right)\right]$$

$$+\left\{K_{14}+\frac{\delta}{\lambda}\left\{\frac{K_{141}\left(K_{132}-\dfrac{K_{133}}{\lambda Be^{-h/\epsilon}+K_{02}}\right)}{\sqrt{1-\left(\lambda Be^{-h/\epsilon}+K_{02}\right)^2}}+\frac{K_{142}\,K_{133}}{\lambda Be^{-h/\epsilon}+K_{02}}\right.\right.$$

$$+ K_{143}\ln\!\left[\tan\!\left(\frac{\pi}{4}+\frac{\cos^{-1}\!\left(\lambda Be^{-h/\epsilon}+K_{02}\right)}{2}\right)\right]$$

$$+\left(K_{132}K_{142}+K_{144}\!\left(\frac{\pi}{2}-K_{15}\right)\right)\!\left[\frac{4\ln\!\left(\lambda Be^{-h/\epsilon}+K_{02}\right)}{3}-\frac{\left(\lambda Be^{-h/\epsilon}+K_{02}\right)^2}{6}\right]$$

$$+ K_{145}\!\left[\frac{\left[\cos^{-1}\!\left(\lambda Be^{-h/\epsilon}+K_{02}\right)\right]^3}{6}+\frac{3\left[\cos^{-1}\!\left(\lambda Be^{-h/\epsilon}+K_{02}\right)\right]^5}{40}\right]$$

$$+ K_{146}\!\left[\frac{\left[\cos^{-1}\!\left(\lambda Be^{-h/\epsilon}+K_{02}\right)\right]^2}{2}+\frac{\left[\cos^{-1}\!\left(\lambda Be^{-h/\epsilon}+K_{02}\right)\right]^4}{8}\right]$$

$$\left.\left.+\frac{K_{02}\,K_{144}\cos^{-1}\!\left(\lambda Be^{-h/\epsilon}+K_{02}\right)}{\sqrt{1-K_{02}{}^2}}\right\}\right\}\epsilon + O(\epsilon^2) \tag{4.111}$$

where the inner expansion constants K_{01}-K_{05} are given in Eqs (4.75)-(4.79) and K_{11}-K_{15} are given in Eqs (4.96)-(4.100).

The above set of five equations form an approximate, analytical solution to the set of five, coupled, first order, non-linear ODEs (Eqs (4.1)-(4.5)) that describe the trajectory of a vehicle entering a non-rotating planetary atmosphere. This solution set is first order accurate with respect to the small perturbation parameter $1/\beta r_s$, which is approximately $1/900$ for earth entry. Since a solution is defined as $Y = Y_0 + Y_1\epsilon + O(\epsilon^2)$ and the orders of Y_0 and Y_1 are similar by definition, addition of the first order solution $Y_1\epsilon$ increases the accuracy of the solution by three orders of magnitude.

Eqs (4.107)-(4.111) are relatively simple and accurate analytical solutions to the complex physical system of atmospheric entry. They provide a readily available analysis which retains the subtleties of the original system. It is this characteristic of analytical solutions which makes them more favorable than obtaining solutions using numerical analyses.

V. Validity of the Solutions to the Equations of Motion and Comparison to Numerical Solutions

In Section IV, analytical solutions are derived solving the set of five, coupled, first order, non-linear ODEs that describe the trajectory of a vehicle entering a non-rotating planetary atmosphere. These solutions, given as Eqs (4.107)-(4.111), are first order accurate to the small perturbation parameter $1/\beta r_s$, which is approximately $1/900$ for earth entry. They provide a relatively simple and accurate solution to the complex, non-linear physical phenomena of atmospheric entry, retaining the trends and subtleties that are lost in an analysis using numerical methods.

Comparison of Analytical and Numerical Results

To demonstrate the first order accurate analytical solution derived in Section IV (Eqs (4.106)-(4.111)), the differential equations of motion (Eqs (4.1)-(4.5)) are numerically integrated and compared to the results from the derived analytical solutions for the same set of initial conditions.

Numerical Approach. To numerically integrate the five equations of motions derived in Section III (Eqs (3.60)-(3.64)), a fourth-order predictor-corrector integrating algorithm is used (Wiesel, 1989:119-123). The integrating step, Δh, is decreased and the equations of motion are repeatedly integrated until the resulting data from the above integration remains unchanged. The numerical integrator given by Wiesel assumes an

85

independent variable that is monotonically increasing or decreasing. For this study, the non-dimensional altitude h is the independent variable, but it is not monotonically decreasing, since the skipping of the re-entry vehicle results in a local oscillation in altitude. Thus, the numerical integrator is slightly modified to account for this altitude oscillation by changing the sign of the altitude increment based on the sign of the flight path angle.($\Delta h > 0$ if $\gamma > 0$ and $\Delta h < 0$ if $\gamma < 0$)

Solution Comparison. To compare the analytical and numerical techniques to solve the differential equations of motion, the initial conditions for an Apollo-type reentry vehicle are used (Hillje, 1969:2-10). Figures F1-F5 (Appendix F) show that the analytical solution give very accurate solutions to the equations of motion. The advantage of having a first order accurate solution is evident in the increased solution accuracy around skip points, where the flight path angle changes sign, and at low altitudes, where aerodynamic forces dominate.

Derivation Assumptions

To derive analytical solutions to the five ODE equations of motion, Eqs (4.1)-(4.5), several assumptions or approximations were made to arrive at analytical solutions. The first approximations used were introduced when the small perturbation expansions for $\sin(a + b\epsilon + c\epsilon^2 + O(\epsilon^3))$ and $\cos(a + b\epsilon + c\epsilon^2 + O(\epsilon^3))$ were performed. Although the results of the expansions were not exact, they were correct to order ϵ^2. Since the final solutions were order ϵ^0 and ϵ^1, this assumption does not effect the zero and

86

first order solutions. This conclusion is also applicable to the binomial expansions used in initial expansion of the equations of motion and the zero and first order matching expansions.

The next approximation used was to determine the small perturbation expansion for $\tan(a + b\epsilon + c\epsilon^2 + O(\epsilon^3))$. Although the expansion is correct to order ϵ^2, like the sine and cosine expansions above, it assumed $\tan(a) \ll 1/\tan(b\epsilon + c\epsilon^2 + O(\epsilon^3))$. Since b and c ~ O(1) and ϵ ~ O(0.001), this is a valid assumption for most common orbits. The exceptions to this inequality are discussed in the next section.

The remaining assumptions used during the derivation of the zero and first order solutions were made to solve the first order, inner expansions. Through numerical analysis, the only significant approximation made was in deriving the q_1 inner solution, where

$$\int \left(\lambda B e^{-\xi} + K_{02}\right) \exp\left[\frac{2}{\lambda}Cos^{-1}\left(\lambda B e^{-\xi} + K_{02}\right)\right] d\xi$$

$$\sim \int \left(\lambda B e^{-\xi} + K_{02}\right) \exp\left[\frac{2}{\lambda}Cos^{-1}\left(K_{02}\right)\right] d\xi \qquad (5.1)$$

Since a 'q_1' term is present in all of the first order ODEs, the error induced by this approximation is present in all of the first order solutions. This is reflected in Figures E3-E6.

Solution Validity and Restrictions

Even before the above assumptions and approximations were used, the solutions to the five equations of motion were restricted due to

87

singularities found in the ODEs. These singularities were present in every ODE except dq/dh and involved terms of $\sec(\gamma)$, $\csc(\gamma)$ and $\csc(I)$. Thus, the solutions to the ODEs, whether numerical or analytical, become numerically unstable at extremely shallow flight path angles, $|\gamma| \ll 1°$ or at extremely steep flight path angles, $\gamma \sim -90°$. The latter entry trajectory is not realistic since it results in enormous aerodynamic heating and decelerating forces. The solutions also encounter instabilities when the entry body is in the equatorial plane, $I \sim 0°$.

In using the small perturbation expansions for the tangent of I and γ, the approximations impose restrictions on the validity of the derived analytical solutions. As mentioned above, the expansions assumed $\tan(I) \ll 1/\tan(I_1\epsilon + I_2\epsilon^2 + O(\epsilon^3))$ and $\tan(\gamma) \ll 1/\tan(\gamma_1\epsilon + \gamma_2\epsilon^2 + O(\epsilon^3))$. The assumptions become invalid when $O(\tan(I)) \sim O(1/\tan(I_1\epsilon + I_2\epsilon^2 + O(\epsilon^3)))$ or $O(\tan(\gamma)) \sim O(1/\tan(\gamma_1\epsilon + \gamma_2\epsilon^2 + O(\epsilon^3)))$. For earth re-entry, where $\epsilon \sim 1/900$, the above assumptions break down when $85° \leq |I|$ (or $|\gamma|$) $\leq 95°$ For martian re-entry, where the mean planetary radius is smaller than the earth's and the atmosphere is thinner than earth's ($\epsilon \sim 1/350$ (Vinh and others, 1980:5)), the above assumptions break down when $80° \leq |I|$ (or $|\gamma|$) $\leq 90°$. Thus, the approximate solutions become invalid at extremely steep entry trajectories or near polar orbits.

The assumption given as Eq (5.1) was used to facilitate the first order, inner solution for q_1. The approximation $\lambda B e^{-\xi} \sim 0$ was found accurate until the entry vehicle penetrated the lowest regions of the atmosphere, when the approximation slightly underestimated the changes in q_1, since as $\xi \to 0$, $e^{-\xi} \sim 1$. Since this approximation is carried forward to the first order, inner expansion ODEs for I_1, Ω_1 and α_1, in the form of the q_1 terms present in each of the ODEs, this underestimation trend is present in

88

all of the above solutions, as shown in Figures E4-E6. The induced errors occur at very low altitudes, where terminal course corrections, such as lift/drag modulation, are performed. But lift/drag modulation entails changing the lift/drag coefficient, which was assumed constant for this study. Thus, the errors induced by the above approximation are de-emphasized.

VI. Conclusions and Recommendations

Conclusions

Using the Method of Matched Asymptotic Expansions, this study developed first order accurate, analytical solutions (Eqs (4.107)-(4.111)) to the five, coupled, non-linear equations of motion describing three-dimensional, planetary atmospheric entry. A non-rotating planet and atmosphere were assumed, as well as a constant lift-to-drag ratio and ballistic coefficient. The validity of the developed solutions are coordinate dependent since singularities are present in the original equations of motion. As a result of this study, the following conclusions are made

1. A computerized, symbolic, algebraic manipulator greatly reduced the workload in generating the zero and first order asymptotic expansions. Application of a similar program could be used in generating similar expansions for any arbitrary set of differential equations.

2. Exact solutions were found for both of the zero order, outer and inner asymptotic expansions and the first order outer asymptotic expansions. To derive analytical solutions for the first order inner asymptotic expansions, approximations were used resulting in underestimation of the aerodynamic turning at low altitudes. This inaccuracy becomes evident at low altitudes where terminal maneuvers are initiated.

Recommendations

Based on the analysis of the assumptions and the results of this study, the following recommendations for further study are proposed.

1. Using Mathematica™ greatly reduced the workload of finding the outer and inner asymptotic expansions to the equations of motion. Development of a generalized version of the Mathematica™ program used in this study would result in a program generating the nth order asymptotic expansions for any arbitrary differential equation.

2. Due to approximations in small perturbations expansions, the application of this study is not valid for entry trajectories near polar orbits. A different set of coordinate transformations and subsequent expansions should be applied to extend the domain of solutions near this orbital state.

3. Due to approximations in the first order, inner expansion solution for q, the flight path angle, the solutions found in this study become inaccurate for low vehicle altitudes. Further investigation should be undertaken to find either exact solutions to the first order, inner expansion ODEs or better approximations which would increase the accuracy of the analytical solutions at low altitudes.

Appendix A:
Transformation of Planetocentric Coordinates to Orbital Elements Using Spherical Trigonometry

This appendix derives the relations between the planetocentric angles from the equations of motion(θ, ϕ and ψ) and the classical orbital elements (α, Ω and I). Since this study models the planet as a sphere (Vinh and others, 1980:2), spherical trigonometry is used to derive relationships between the planetocentric angles and the orbital elements (Bain, 1989).

Figure A1 shows the relationship between the two related variable sets in an osculating orbit (Vinh and others, 1980:257). For clarity, Figure A2 shows the pertinent variables as a spherical triangle formed by the arcs of three great circles on the surface of a sphere (planet). By definition, the interior angles are angles between the curved line segments and the exterior angles are the angles between the linear segments emanating from the origin, point O. The following is a list of the interior and exterior angles used in the subsequent derivation.

Exterior Angles:	$a - \theta - \Omega$	$b - \phi$	$c - \alpha$
Interior Angles:	$A - \pi/2 - \psi$	$B - I$	$C - \pi/2$

To relate the above two variable sets, the two fundamental formulas of spherical trigonometry, the law of sines and the law of cosines, are used (Fitzpatrick, 1970:118-120). While the law of sines relates the ratios of interior and opposing exterior angles, the law of cosines relates the cosine of
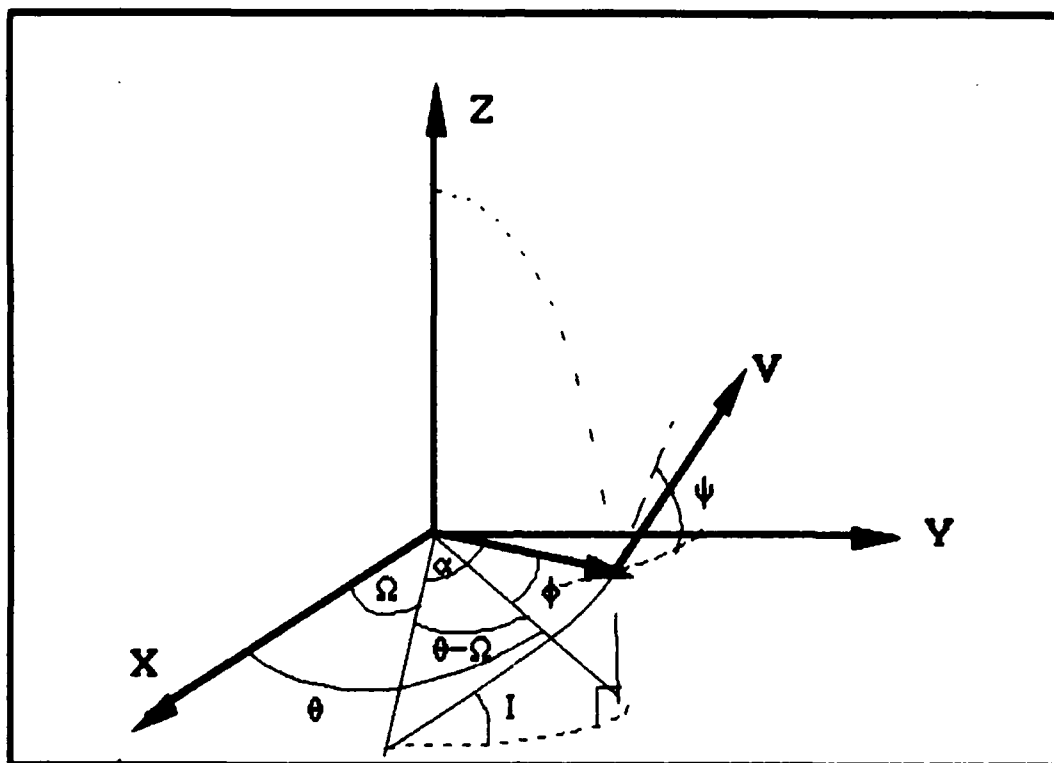
Figure A1. Reference Coordinate System and Orbital
Elements

an interior angle as the sum of the products of the cosines of the other two
interior angles and the products of the sines of the other two interior angles
with the cosine of the opposing exterior angle.

Law of Sines: $\dfrac{\sin(a)}{\sin(A)} = \dfrac{\sin(b)}{\sin(B)} = \dfrac{\sin(c)}{\sin(C)}$ (A.1)

Law of Cosines: $\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(A)$ (A.2)

To derive the three spherical trigonometric relationships, known
values are first substituted into the law of sines

$\dfrac{\sin(\theta - \Omega)}{\sin(\pi/2 - \psi)} = \dfrac{\sin(\phi)}{\sin(I)} = \dfrac{\sin(\alpha)}{\sin(\pi/2)}$ (A.3)
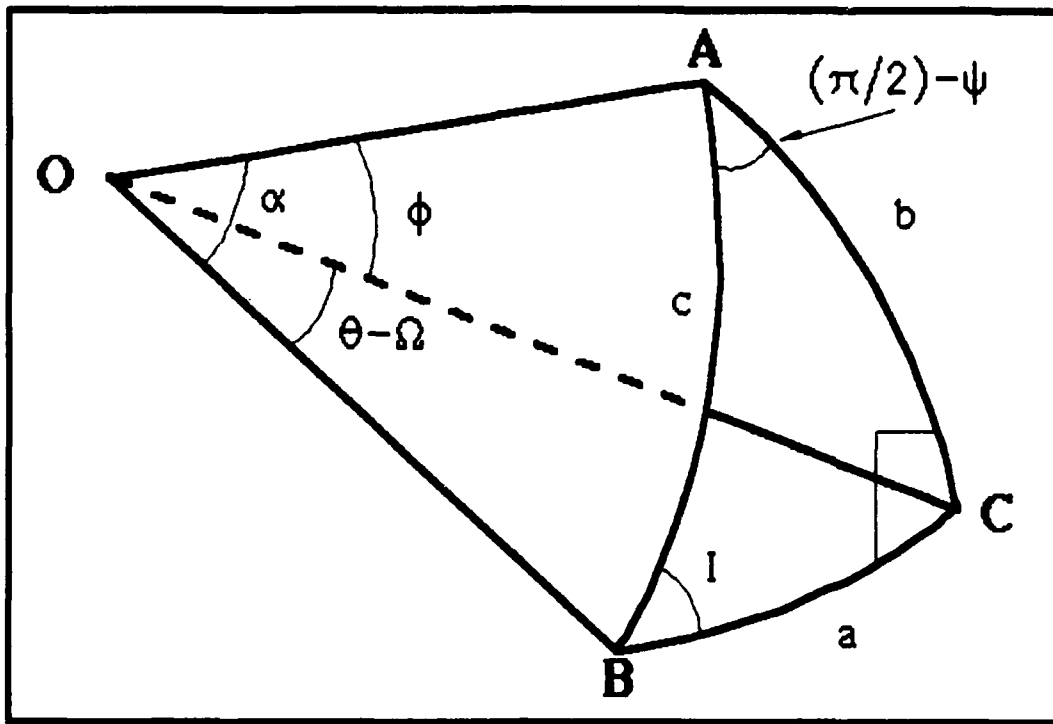
Figure A2.   Spherical Triangle Relating
Planetocentric Angles to Orbital Elements

Since $\sin(\pi/2 - \psi) = \cos(\psi)$ and $\sin(\pi/2) = 1$, two relations arise from Eq (A.3) and making use of the common trigonometric relationship, $\sin^2(x) + \cos^2(x) = 1$:

$$\sin(\alpha) = \frac{\sin(\phi)}{\sin(I)} \qquad \text{or} \qquad \cos^2(\alpha) = 1 - \frac{\sin^2(\phi)}{\sin^2(I)} \qquad (A.4)$$

$$\sin(\theta - \Omega) = \frac{\sin(\phi)\cos(\psi)}{\sin(I)} \qquad \text{or} \qquad \cos^2(\theta - \Omega) = 1 - \frac{\sin^2(\phi)\cos^2(\psi)}{\sin^2(I)} \qquad (A.5)$$

Using the law of cosines gives the first relationship as

$$\cos(\alpha) = \cos(\theta - \Omega)\cos(\phi) + \sin(\theta - \Omega)\sin(\phi)\cos(\pi/2) \quad \text{or}$$

$$\cos(\alpha) = \cos(\theta - \Omega)\cos(\phi) \qquad (A.6)$$

94

Squaring (A.6) and substituting Eqs (A.4) and (A.5) into (A.6) gives

$$1 - \frac{\sin^2(\phi)}{\sin^2(I)} = \left(1 - \frac{\sin^2(\phi)\cos^2(\psi)}{\sin^2(I)}\right)\cos^2(\phi)$$

Simplifying the above expression gives the second relationship:

$$\cos(I) = \pm\cos(\psi)\cos(\phi) \tag{A.7}$$

To derive the third relationship, Eq (A.3) also gives

$$\sin(\theta - \Omega) = \frac{\cos(\psi)\sin(\phi)}{\sin(I)} \tag{A.8}$$

But from Eq (A.7)

$$\cos(\psi) = \frac{\cos(I)}{\cos(\phi)} \tag{A.9}$$

Substituting Eq (A.9) into Eq (A.8) gives the third relationship:

$$\sin(\theta - \Omega) = \frac{\tan(\phi)}{\tan(I)} \tag{A.10}$$

Thus Eqs (A.6), (A.7) and (A.10) give the three spherical trigonometric relationships required to transform the planetocentric angles to classical orbital elements. For these equations to be useful, they will be reformed to match the equations of motion Eqs (3.46)-(3.50). Eq (A.4) is already in the correct format. Repeating Eq (A.6)

$$\cos(\theta - \Omega) = \frac{\cos(\alpha)}{\cos(\phi)} \tag{A.11}$$

But from the law of cosines

$$\cos(\theta - \Omega) = \cos(\phi)\cos(\alpha) + \sin(\phi)\sin(\alpha)\cos(\pi/2 - \psi)$$

$$= \cos(\phi)\cos(\alpha) + \sin(\phi)\sin(\alpha)\sin(\psi) \qquad (A.12)$$

Equating Eqs (A.11) and (A.12) and simplifying gives

$$\sin(\psi) = \frac{\cos(\alpha)\sin(I)}{\cos(\phi)} \qquad (A.13)$$

Substituting Eq (A.4) into Eq (A.13) gives

$$\sin(\psi) = \frac{\tan(\phi)}{\tan(\alpha)} \qquad (A.14)$$

Again taking the law of cosines

$$\cos(\pi/2 - \psi) = \cos(I)\cos(\pi/2) + \sin(I)\sin(\pi/2)\cos(\theta - \Omega) \quad \text{or}$$

$$\sin(\psi) \qquad = \sin(I)\cos(\theta - \Omega) \qquad (A.15)$$

To summarize, the three transformation relationships between planetocentric angles (ϕ, ψ and θ) and classical orbital elements (α, Ω and I) are:

$$\sin(\phi) = \sin(\alpha)\sin(I) \qquad (A.4)$$

$$\cos(I) = \cos(\psi)\cos(\phi) \qquad (A.7)$$

$$\sin(\psi) = \sin(I)\cos(\theta - \Omega) \qquad (A.15)$$

Appendix B:
Mathematica™ Code to Generate Zero, First, and Second Order Asymptotic Expansions

This appendix lists the Mathematics™ code used to generate the zero, first and second order outer and inner expansions to the five equations of motion

Program Structure

The Mathematica™ program is structured to input a matrix whose first column is composed of the number of ODEs, followed by the ODEs themselves. The program substitutes the small perturbation expansions for the dependent variables, Eq (4.6) and also for trigonometric and algebraic functions of the dependent variables. For the outer expansions, the program multiplies out all of the terms composed of sums of constant coefficients and powers of ϵ. The program then combines the coefficients of ϵ^0, ϵ^1 and ϵ^2. The ϵ^0 terms are the zero order outer expansion ODEs, ϵ^1 terms are the first order outer expansion ODEs and ϵ^2 terms are the second order outer expansion ODEs. To perform the inner expansions, the independent variable h is changed to the magnified variable ξ by the stretching transformation $h - \xi\epsilon$. The program then multiplies and collects terms as in the outer expansions just completed. The ϵ^0 terms are the zero order

inner expansion ODEs, ϵ^1 terms are the first order inner expansion ODEs and ϵ^2 terms are the second order inner expansion ODEs.

The core of the program is the sequence which multiplies out the small perturbation expansions of the dependent and independent variables and collects coefficients of powers of ϵ. These steps are executed by the Mathematica™ functions Expand[] and Coefficient[]. Expand[] writes the products of polynomials, in powers of ϵ, as a simple sum of terms of constants coefficients and powers of ϵ with all products expanded out. Coefficient[] collects coefficients of a prescribed power of ϵ from the the above sum of terms (Wolfram, 1988:381-384).

Program Listing

```
*****************************************************************
Expansions : : usage - "This Mathematica function gives the zero, first
        and second outer and inner asymptotic expansions for a set of first
        order differential equations of motion (EOM).  The EOM are
        inputted via a matrix called 'ode'.  ode[[1,1]] is the number of EOM
        being expanded.  ode[[2,1]], ode[[3,1]], etc are the actual EOM.  The
        expansion parameter must be called 'e' for the program to work."


Expansions[ode_] : -


(* Defining small perturbation expansions needed for outer and inner
        expansions *)


(* Defining the small perturbation expansions for the dependent
        variables in terms of the small parameter, e *)
u     - u0    + u1*e     + u2*e^2 ;
q     - q0    + q1*e     + q2*e^2 ;
```

```
i     = i0    + i1*e    + i2*e^2 ;
o     = o0    + o1*e    + o2*e^2 ;
a     = a0    + a1*e    + a2*e^2 ;
g     = g0    + g1*e    + g2*e^2 ;
```

(* Defining the small perturbation expansion for trigonometric
    functions *)
Sin[x0_ + x1_*e_ + x2_*e_^2] = Sin[x0] + x1*Cos[x0]*e + ((x2*Cos[x0]) -
    (x1^2*sin(x0)/2))*e^2 ;

Cos[x0_ + x1_*e_ + x2_*e_^2] = Cos[x0] - x1*Sin[x0]*e - ((x2*Sin[x0]) +
    (x1^2*cos(x0)/2))*e^2 ;

Tan[x0_ + x1_*e_ + x2_*e_^2] = Tan[x0] + (x1/Cos[x0]^2)*e +
    ((x2/Cos[x0]^2) + (x1^2*sin(x0)/cos(x0)^3))*e^2 ;

(* Defining the small perturbation expansion for the reciprocal of
    trigonometric functions *)
Sin[x0_ + x1_*e_ + x2_*e_^2]^-1 = (1/Sin[x0]) -
    (x!*Cos[x0]/Sin[x0]^2)*e + ((x1^2*(1 + Cos[x0]^2)/2*Sin[x0]^3) -
    (x2*Cos[x0]/Sin[x0]^2))*e^2 ;

Cos[x0_ + x1_*e_ + x2_*e_^2]^-1 = (1/Cos[x0]) +
    (x1*Sin[x0]/Cos[x0]^2)*e + ((x1^2*(1 + Sin[x0]^2)/2*Cos[x0]^3) +
    (x2*Sin[x0]/Cos[x0]^2))*e^2 ;

Tan[x0_ + x1_*e_ + x2_*e_^2]^-1 = (1/Tan[x0]) - (x1/Sin[x0]^2)*e +
    ((x1^2*Cos[x0]^2/Sin[x0]^3) - (x2/Sin[x0]^2))*e^2 ;

(* Defining the small perturbation expansion for algebraic functions *)
(x0_ + x1_*e_ + x2_*e_^2)^-1 = (1/x0) - (x1/x0^2)*e + ((x1^2/x0^3) -
    (x2/x0^2))*e^2 ;

(* Defining the exponential of a large negative number is zero, or
    allowing the expansion to neglect exponentially small terms *)
(E^(h/e))^-1 = 0 ;

99

(* Performing the outer expansions of the EOM *)

(* Multiplying out all the terms in the EOM *)
Do[ode[[i, 2]] = Expand[ode[[i, 1]]], {i, 2, (1 + ode[[1 , 1]])}] ;
i = . ;

(* Grouping the coefficients of e raised to the 0,1 and 2 powers. The
    zero order outer expansions are located in ode[[2 , 3]]-ode[[6 , 3]],
    the first order outer expansions are located in ode[[2 , 4]]-ode[[6 ,
    4]], and the second order outer expansions are located in ode[[2 ,
    5]]-ode[[6 , 5]]. *)
Do[ode[[i, 3]] = Coefficient[ode[[i, 2]], e , 0], {i, 2, (1 + ode[[1 , 1]])}]
i = . ;
Do[ode[[i, 4]] = Coefficient[ode[[i, 2]], e , 1], {i, 2, (1 + ode[[1 , 1]])}]
i = . ;
Do[ode[[i, 5]] = Coefficient[ode[[i, 2]], e , 2], {i, 2, (1 + ode[[1 , 1]])}]
i = . ;

(* Performing the inner expansions of the EOM, which transforms h
    to z by the definition h = z*e)

(* Multiplying out all the terms in the EOM *)
Do[ode[[i, 2]] = e*ode[[i, 1]], {i, 2, (1 + ode[[1 , 1]])}] ;
i = . ;

(* Implementing the definition relating h and the stretching variable z
     *)
h = z*e ;

(* Multiplying out all the terms in the EOM *)
Do[ode[[i, 2]] = Expand[ode[[i, 2]]], {i, 2, (1 + ode[[1 , 1]])}] ;
i = . ;

100

(* Grouping the coefficients of e raised to the 0,1 and 2 powers. The
   zero order inner expansions are located in ode[[2,6]]-ode[[6,6]],
   the first order inner expansions are located in ode[[2,6]]-ode[[6,
   6]], and the second order inner expansions are located in ode[[2,
   6]]-ode[[6,6]]. *)

Do[ode[[i,6]] = Coefficient[ode[[i,2]],e,0],{i,2,(1 + ode[[1,1]])}]

i = . ;

Do[ode[[i,7]] = Coefficient[ode[[i,2]],e,1],{i,2,(1 + ode[[1,1]])}]

i = . ;

Do[ode[[i,8]] = Coefficient[ode[[i,2]],e,2],{i,2,(1 + ode[[1,1]])}]

i = . ;

****

Sample Input

The following listing is a sample input required to execute the above
program. It is same the five ODEs derived in Section IV and describe the
flight trajectory of a lifting body entering a planetary atmosphere.

```
ode[[1,1]] = 5
ode[[2,1]] = u/(1 + h) - (2*b*u*(1 + l*Tan[g]))/(E^(h/e)*e*Sin[g])
ode[[3,1]] = -((b*l)/(E^(h/e)*e)-(q*(1 - q^2/u))/(1 + h)
ode[[4,1]] = 1/((1 + h)*Tan[g]) - (b*d*Sin[a])/(E^(h/e)*e*Cos[g]*Sin[g]*Tan[i])
ode[[5,1]] = (b*d*Sin[a])/(E^(h/e)*e*Cos[g]*Sin[g]*Sin[i])
ode[[6,1]] = (b*d*Cos[a])/(E^(h/e)*e*Cos[g]*Sin[g])
```

Appendix C:
Derivation of Frequently Used Taylor Series and Binomial Expansions

This appendix derives many Taylor series and binomial expansions frequently used in this study.

Small Perturbation Expansions for Trigonometric Functions

To expand the sine, cosine and tangent functions of small perturbations (powers of ϵ) into linear combinations of powers of the perturbations, *the small angle formulas for the above functions are needed.* The small perturbation here is ϵ, where $\epsilon \ll 1$

Since cos (x) is defined as

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{(2n)!}$$

$$\cos(a\epsilon) = 1 - \frac{(a\epsilon)^2}{2!} + \frac{(a\epsilon)^4}{4!} + ...$$

$$= 1 + \frac{a^2}{2!} \epsilon^2 + O(\epsilon^4) \qquad\qquad (C.1)$$

Since sin(x) is defined as

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)!}$$

$$\sin(a\epsilon) = a\epsilon - \frac{(a\epsilon)^3}{3!} + \frac{(a\epsilon)^5}{5!} + \dots$$

$$= a\epsilon + O(\epsilon^3) \qquad (C.2)$$

Since $\tan(x)$ is defined as (Beyer, 1984:297) and (Gradshteyn and Ryzhik, 1980:34-35), where B_{2k} is the nth Bernoulli number

$$\tan(x) = \sum_{n=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|\, x^{2k-1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{2x^5}{15!} + \dots$$

$$\tan(a\epsilon) = a\epsilon + \frac{(a\epsilon)^3}{3!} + \frac{2(a\epsilon)^5}{15!} + \dots$$

$$= a\epsilon + O(\epsilon^3) \qquad (C.3)$$

To derive the small perturbation expansions, angle-sum relations are also needed.

$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b) \qquad (C.4)$$

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad (C.5)$$

$$\tan(a + b) = \frac{\tan(a)\tan(b)}{1 - \tan(a)\tan(b)} \qquad (C.6)$$

In deriving the small perturbation expansions for $\sin(a + b\epsilon + c\epsilon^2 + d\epsilon^3)$ and $\cos(a + b\epsilon + c\epsilon^2 + d\epsilon^3)$, Eqs (C.4) and (C.5) are repeatedly used. The following example illustrates this by deriving the expansion for $\sin(a + b\epsilon + c\epsilon^2 + d\epsilon^3)$.

$$\sin(a + b\epsilon + c\epsilon^2 + d\epsilon^3)$$

$$= \sin(a)\cos(b\epsilon + c\epsilon^2 + d\epsilon^3) + \cos(a)\sin(b\epsilon + c\epsilon^2 + d\epsilon^3)$$

103

$$- \sin(a) \left[\cos(b\epsilon)\cos(c\epsilon^2 + d\epsilon^3) - \sin(b\epsilon)\sin(c\epsilon^2 + d\epsilon^3)\right]$$
$$+ \cos(a) \left[\sin(b\epsilon) \cos(c\epsilon^2 + d\epsilon^3) + \cos(b\epsilon) \sin(c\epsilon^2 + d\epsilon^3)\right]$$

$$= \sin(a) \left\{\cos(b\epsilon)[\cos(c\epsilon^2) \cos(d\epsilon^3) - \sin(c\epsilon^2) \sin(d\epsilon^3)]\right.$$
$$- \sin(b\epsilon) \left[\sin(c\epsilon^2) \cos(d\epsilon^3) + \cos(c\epsilon^2) \sin(d\epsilon^3)\right]\}$$
$$+ \cos(a) \left\{\sin(b\epsilon) [\cos(c\epsilon^2) \cos(d\epsilon^3) - \sin(c\epsilon^2) \sin(d\epsilon^3)]\right.$$
$$+ \cos(b\epsilon)[\sin(c\epsilon^2) \cos(d\epsilon^3) + \cos(c\epsilon^2) \sin(d\epsilon^3)]\}$$

Substituting in Eqs (C.1) and (C.2)

$$\sin(a + b\epsilon + c\epsilon^2 + d\epsilon^3) = \sin(a) + b\cos(a)\epsilon$$

$$+ \left(c\cos(a) - \frac{b^2\sin(a)}{2}\right)\epsilon^2 + O(\epsilon^3) \qquad (C.7)$$

Similarly, $\cos(a + b\epsilon + c\epsilon^2 + d\epsilon^3)$ becomes

$$\cos(a + b\epsilon + c\epsilon^2 + d\epsilon^3) = \cos(a) - b\sin(a)\epsilon$$

$$- \left(c\sin(a) + \frac{b^2\cos(a)}{2}\right)\epsilon^2 + O(\epsilon^3) \qquad (C.8)$$

To derive the expressions for the reciprocals of the above relationships, binomial coefficients and Pochhammer symbols are used (Andrews, 1985:10-11, 273). By definition, where $(-n)_0 = 1$ and $(-n)_k = (-n) (-n +1)...(-n + k - 1)$

$$(a + b)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^{n-k} b^k = \sum_{k=0}^{\infty} \frac{(-1)^k (-n)_k}{k!} a^{n-k} b^k$$

$$(C.9)$$

As an example to find the reciprocal of a binomial series

$$\frac{1}{a + b\epsilon + c\epsilon^2 + d\epsilon^3} = \sum_{k=0}^{\infty} \binom{-1}{k} a^{-1-k} \left(b\epsilon + c\epsilon^2 + d\epsilon^3\right)^k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (1)_k}{k!} a^{-(1+k)} \left(b\epsilon + c\epsilon^2 + d\epsilon^3\right)^k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{a^{(1+k)}} \left(b\epsilon + c\epsilon^2 + d\epsilon^3\right)^k$$

$$= \frac{1}{a} - \frac{b}{a^2}\epsilon + \left(\frac{b^2}{a^3} - \frac{c}{a^2}\right)\epsilon^2 + O(\epsilon^3) \qquad \text{(C.10)}$$

Substituting Eqs (C.7) and (C.8) into (C.10) gives

$$(\sin(a + b\epsilon + c\epsilon^2 + d\epsilon^3))^{-1}$$

$$= \frac{1}{\sin(a)} - \frac{b\cos(a)}{\sin^2(a)}\epsilon + \left(\frac{b^2\left(1 + \cos^2(a)\right)}{2\sin^3(a)} - \frac{c\cos(a)}{\sin^2(a)}\right)\epsilon^2 + O(\epsilon^3) \quad \text{(C.11)}$$

$$(\cos(a + b\epsilon + c\epsilon^2 + d\epsilon^3))^{-1}$$

$$= \frac{1}{\cos(a)} + \frac{b\sin(a)}{\cos^2(a)}\epsilon + \left(\frac{b^2\left(1 + \sin^2(a)\right)}{2\cos^3(a)} + \frac{c\sin(a)}{\cos^2(a)}\right)\epsilon^2 + O(\epsilon^3) \quad \text{(C.12)}$$

To derive the small perturbation expansion for $\tan(a + b\epsilon + c\epsilon^2 + d\epsilon^3)$, Eq (C.6) can be used repeatedly, with the assumption $\tan(a) \tan(b\epsilon + c\epsilon^2 + d\epsilon^3) \approx 0$ (alternatively, $\tan(a) \ll 1/\tan(b\epsilon + c\epsilon^2 + d\epsilon^3)$). This assumption worsens as $a \rightarrow \pi/2$ radians or as ϵ increases. Due to this approximation, Karasopoulos limited his study to $I \leq 75°$ (Karasopoulos, 1988:6) To avoid

105

this approximation, Eq (C.7) is divided by Eq (C.12), since $\tan(x) =$ $\sin(x)/\cos(x)$.

$$\tan(a + b\epsilon + c\epsilon^2 + d\epsilon^3) = \frac{\sin(a + b\epsilon + c\epsilon^2 + d\epsilon^3)}{\cos(a + b\epsilon + c\epsilon^2 + d\epsilon^3)}$$

$$= \left[\sin(a) + b\cos(a)\epsilon + \left(c\cos(a) - \frac{b^2\sin(a)}{2}\right)\epsilon^2\right]$$

$$\times \left[\frac{1}{\cos(a)} + \frac{b\sin(a)}{\cos^2(a)}\epsilon + \left(\frac{b^2\left(1 + \sin^2(a)\right)}{2\cos^3(a)} + \frac{c\sin(a)}{\cos^2(a)}\right)\epsilon^2\right] + O(\epsilon^3)$$

Simplifying and grouping terms of powers of ϵ gives

$$\tan(a + b\epsilon + c\epsilon^2 + d\epsilon^3)$$

$$= \tan(a) + \frac{b}{\cos^2(a)}\epsilon + \left(\frac{b^2\sin(a)}{\cos^3(a)} + \frac{c}{\cos^2(a)}\right)\epsilon^2 + O(\epsilon^3) \qquad (C.13)$$

To derive the reciprocal of the above expression, Eq (C.8) is divided by Eq (C.11), since $1/\tan(x) = \cos(x)/\sin(x)$.

$$(\tan(a + b\epsilon + c\epsilon^2 + d\epsilon^3))^{-1} = \frac{\cos(a + b\epsilon + c\epsilon^2 + d\epsilon^3)}{\sin(a + b\epsilon + c\epsilon^2 + d\epsilon^3)}$$

$$= \left[\cos(a) - b\cos(a)\epsilon - \left(c\sin(a) + \frac{b^2\cos(a)}{2}\right)\epsilon^2\right]$$

$$\times \left[\frac{1}{\sin(a)} + \frac{b\cos(a)}{\sin^2(a)}\epsilon + \left(\frac{b^2\left(1 + \cos^2(a)\right)}{2\sin^3(a)} - \frac{c\cos(a)}{\sin^2(a)}\right)\epsilon^2\right] + O(\epsilon^3)$$

Simplifying and grouping terms of powers of ϵ gives

$$(\tan(a + b\epsilon + c\epsilon^2 + d\epsilon^3))^{-1}$$

106

$$= \frac{1}{\tan(a)} - \frac{b}{\sin^2(a)}\epsilon + \left(\frac{b^2\cos(a)}{\sin^3(a)} - \frac{c}{\sin^2(a)}\right)\epsilon^2 + O(\epsilon^3) \qquad (C.14)$$

Small Perturbation Expansions for Algebraic Functions

This section derives many of the frequently used binomial expansions used in the derivations in this study. Again binomial coefficients and Pochhammer symbols are used and are defined in Eq (C.9).

$$\frac{1}{\left(a + b\epsilon + c\epsilon^2 + d\epsilon^3\right)^n} = \sum_{k=0}^{\infty} \binom{-n}{k} a^{-n-k} \left(b\epsilon + c\epsilon^2 + d\epsilon^3\right)^k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (n)_k}{k!} a^{-(n+k)} \left(b\epsilon + c\epsilon^2 + d\epsilon^3\right)^k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (n)_k}{a^{(n+k)}k!} \left(b\epsilon + c\epsilon^2 + d\epsilon^3\right)^k \qquad (C.15)$$

For example, as shown above in Eq (C.10), for n - 1

$$\frac{1}{(a + b\epsilon + c\epsilon^2 + d\epsilon^3)^1} = \frac{(1)_0 (b\epsilon + c\epsilon^2 + d\epsilon^3)^0}{(0!) a^1}$$

$$- \frac{(1)_1 (b\epsilon + c\epsilon^2 + d\epsilon^3)^1}{(1!) a^2} + \frac{(1)_2 (b\epsilon + c\epsilon^2 + d\epsilon^3)^2}{(2!) a^3} + O(\epsilon^3)$$

$$= \frac{1}{a} - \frac{b\epsilon + c\epsilon^2 + d\epsilon^3}{a^2} + \frac{2(b\epsilon + c\epsilon^2 + d\epsilon^3)^2}{2a^3} + O(\epsilon^3)$$

$$= \frac{1}{a} - \frac{b}{a^2}\epsilon + \left(\frac{b^2}{a^3} - \frac{c}{a^2}\right)\epsilon^2 + O(\epsilon^3) \tag{C.16}$$

As another example, as shown above in Eq (C.16), for $n = \frac{1}{2}$

$$\frac{1}{(a + b\epsilon + c\epsilon^2 + d\epsilon^3)^{\frac{1}{2}}} = \frac{(1)_0\,(b\epsilon + c\epsilon^2 + d\epsilon^3)^0}{(0!)\,a^{\frac{1}{2}}}$$

$$- \frac{\left(\frac{1}{2}\right)_1 (b\epsilon + c\epsilon^2 + d\epsilon^3)^1}{(1!)\,a^{\frac{3}{2}}} + \frac{\left(\frac{1}{2}\right)_2 (b\epsilon + c\epsilon^2 + d\epsilon^3)^2}{(2!)\,a^{\frac{5}{2}}} + O(\epsilon^3)$$

$$= \frac{1}{a^{\frac{1}{2}}} - \frac{b\epsilon + c\epsilon^2 + d\epsilon^3}{2a^{\frac{3}{2}}} + \frac{3(b\epsilon + c\epsilon^2 + d\epsilon^3)^2}{8a^{\frac{5}{2}}} + O(\epsilon^3)$$

$$= \frac{1}{a^{\frac{1}{2}}} - \frac{b}{2a^{\frac{3}{2}}}\epsilon + \left(\frac{3b^2}{8a^{\frac{5}{2}}} - \frac{c}{2a^{\frac{3}{2}}}\right)\epsilon^2 + O(\epsilon^3) \tag{C.17}$$

Appendix D:
Derivation of Solutions for Outer Expansions of the Equations of Motion

Methods of Solution

The solutions to the two sets of five, coupled, first order, linear nonhomogeneous ordinary differential equations (ODEs) found in this appendix are derived below. Three methods are used to solve these equations. The first method used is the Method of Separation of Variables, where the ODE and its solution are given as (Beyer, 1984:315)

$$\frac{dy}{dx} - \frac{f(x)}{f(y)} \rightarrow \int f(y) \, dy + \int f(x) \, dx - C \qquad (D.1)$$

C is the constant of integration.

The second method used to solve ODEs found in this study is the Method of Integrating Factors, where the ODE has the form

$$\frac{dy}{dx} + P(x)y - Q(x) \qquad (D.2)$$

The integrating factor has the form

$$\upsilon - e^{\int P(x) \, dx} \qquad (D.3)$$

and the solution is given as (Beyer, 1984:315)

$$ye^{\int P(x) \, dx} - \int Q(x) \, e^{\int P(x) \, dx} + C \quad \text{or}$$

$$y = \frac{\int Q(x) \, e^{\int P(x) \, dx} + C}{e^{\int P(x) \, dx}} \tag{D.4}$$

The third and final method is a variation of the second method and is known as Bernoulli's Equation (Beyer, 1984:315). The ODE is given as

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{D.5}$$

Again, the integrating factor has the form

$$\upsilon = e^{\int P(x) \, dx} \tag{D.6}$$

and the solution is given, for $n = 1$, as (Beyer, 1984:315)

$$y^{(1-n)}e^{(1-n)\int P(x) \, dx} = (1 - n)\int Q(x) \, e^{(1-n)\int P(x) \, dx} + C \quad \text{or}$$

$$y = \left(\frac{(1 - n)\int Q(x) \, e^{(1-n)\int P(x) \, dx} + C}{e^{(1-n)\int P(x) \, dx}} \right)^{\frac{1}{1-n}} \tag{D.7}$$

Although some of the ODEs are coupled, the coupling is avoided by solving by ODEs in a judicious order. The du/dh ODE is solved first, followed by the dq/dh ODE is second and finally the dα/dh ODE is solved. The dI/dh and dΩ/dh ODEs are independent of the other ODEs and are solved in the order presented for consistency.

Constants of Integration Notation

110

As shown in Section IV, there are ten ODEs expanded from the equations of motion. Accompanying the solution of each of these ODEs is a constant of integration. To simplify the bookkeeping involved in defining all these constants, the following notation is used for the outer expansion solutions:

The constant of integration is given as C_{ij}, where i denotes the order $(\epsilon^i -- \epsilon^0 \text{ or } \epsilon^1)$ of the solution and j denotes the variable associated with this constant.

j	variable
1	u
2	q
3	I
4	Ω
5	α

Thus C_{01} is the constant of integration for the variable u_0 and C_{14} is the constant integration for the variable Ω_1.

Outer Expansion ϵ^0 Terms

The differential equations corresponding to the ϵ^0 equation of motion terms were derived in Section IV and are repeated below:

$$\frac{du_0}{dh} = -\frac{u_0}{1+h} \tag{D.8}$$

$$\frac{dq_0}{dh} = \frac{q_0}{1+h}\left(\frac{q_0^2}{u_0} - 1\right) \tag{D.9}$$

$$\frac{dI_0}{dh} = 0 \tag{D.10}$$

$$\frac{d\Omega_0}{dh} = 0 \tag{D.11}$$

111

$$\frac{d\alpha_0}{dh} = \frac{1}{(1 + h) \tan(\gamma_0)} \qquad (D.12)$$

<u>du_0/dh Equation Solution</u>. The Method of Separation of Variables solves Eq (D.8). Rearranging Eq (D.8) into form of Eq (D.1) gives

$$\frac{du_0}{dh} = -\frac{u_0}{1 + h} \Rightarrow \frac{du_0}{u_0} = -\frac{dh}{1 + h} \Rightarrow \int \frac{du_0}{u_0} du + \int \frac{dh}{1 + h} dh = \tilde{C}_{01}$$

Solving the above ODE gives

$$\ln(u_0) + \ln(1 + h) = \tilde{C}_{01} \quad \text{or} \quad u_0 = \frac{C_{01}}{1 + h}$$

$$\text{Solution: } u_0 = \frac{C_{01}}{1 + h} \qquad (D.13)$$

<u>dq_0/dh Equation Solution</u>. This ODE is solved by substituting the solution of u_0, Eq (D.13), into Eq (D.9). Thus, the ODE now becomes

$$\frac{dq_0}{dh} = \frac{q_0}{1 + h}\left(\frac{q_0^2}{u_0} - 1\right)$$

$$= \frac{q_0}{1 + h}\left(\frac{q_0^2}{\frac{C_{01}}{1 + h}} - 1\right)$$

$$= \frac{q_0^3}{C_{01}} - \frac{q_0}{1 + h}$$

The ODE now has the form of Bernoulli's equation Eq (D.5), where

$$P(x) = \frac{1}{1 + h} \qquad Q(x) = \frac{1}{C_{01}} \qquad n = 3$$

From Eq (D.6)

$$\upsilon = e \int P(x)\, dx = e \int \frac{1}{1+h}\, dh = \ln(1+h)$$

Thus, Eq (D.7) gives

$$q_0^{-2} e^{-2\ln(1+h)} = (-2)\int \frac{1}{C_{01}} e^{-2\ln(1+h)} dh - C_{02} \quad \text{or}$$

$$\frac{1}{\left(q_0(1+h)\right)^2} = \frac{-2}{C_{01}} \int \frac{1}{(1+h)^2}\, dh - C_{02}$$

Integrating the right hand side of the above equation gives

$$\frac{1}{\left(q_0(1+h)\right)^2} = \frac{2}{C_{01}(1+h)} - C_{02}$$

Solving for q_0 gives the desired solution.

$$\text{Solution:} \quad q_0 = \frac{1}{\sqrt{\frac{2}{C_{01}}(1+h) - C_{02}(1+h)^2}}$$

$$= \frac{C_{01}\sqrt{C_{02}}}{\sqrt{1 - \left[C_{01}C_{02}(1+h) - 1\right]^2}} \tag{D.14}$$

$\underline{dI_0/dh\ \text{Equation Solution}}$. The solution to this ODE (Eq (D.10)) is trivial.

$$\text{Solution:} \quad I_0 = C_{03} \tag{D.15}$$

$\underline{d\Omega_0/dh\ \text{Equation Solution}}$. The solution to this ODE (Eq (D.11)) is trivial.

$$\text{Solution:} \quad \Omega_0 = C_{04} \tag{D.16}$$

$d\alpha_0/dh$ Equation Solution. This ODE is solved by first rewriting the term $\tan(\gamma_0)$ in Eq (D.12) in terms of q_0, where, by definition, $q_0 = \cos(\gamma_0)$.

$$\frac{1}{\tan(\gamma_0)} = \frac{1}{\dfrac{\sin(\gamma_0)}{\cos(\gamma_0)}} = \frac{1}{\dfrac{\sqrt{1 - \cos^2(\gamma_0)}}{\cos(\gamma_0)}} = \frac{1}{\dfrac{\sqrt{1 - q_0^2}}{q_0}} = \frac{1}{\sqrt{\dfrac{1}{q_0^2} - 1}}$$

But from Eq (D.14)

$$\frac{1}{q_0^2} = \frac{2}{C_{01}}(1 + h) - C_{02}(1 + h)^2$$

Substituting the above two relations into the ODE, Eq (D.12) gives

$$\frac{d\alpha_0}{dh} = \frac{1}{(1 + h)} \frac{1}{\sqrt{\dfrac{2}{C_{01}}(1 + h) - C_{02}(1 + h)^2 - 1}}$$

The Method of Separation of Variables is used to solve this ODE. From Eq (D.1), the solution has the form

$$\int d\alpha_0 = \int \frac{1}{(1 + h)} \frac{1}{\sqrt{\dfrac{2}{C_{01}}(1 + h) - C_{02}(1 + h)^2 - 1}} \, dh + C_{05}$$

The above integral has the following form and solution (Beyer, 1984:257),.where $x = 1 + h$, $dx = dh$, $a = -1$, $b = 2/C_{01}$ and $c = -C_{02}$.

$$\int \frac{1}{x\sqrt{a + bx + cx^2}} \, dx = \frac{1}{\sqrt{-a}} \operatorname{Sin}^{-1}\left(\frac{bx + 2a}{|x|\sqrt{b^2 - 4ac}}\right) + C$$

From the definition of h (Eq (3.22)), $h \geq 0$, therefore $|1 + h| = 1 + h$. Using this relation and substituting the above relation into the ODE gives the solution

114

$$\alpha_0 = \frac{1}{\sqrt{1}} \, \text{Sin}^{-1} \left[\frac{\frac{2(1+h)}{C_{01}} - 2}{(1+h) \sqrt{\left(\frac{2}{C_{01}}\right)^2 - 4C_{02}}} \right] + C_{05}$$

Simplifying the above equation gives

$$\text{Solution: } \alpha_0 = \text{Sin}^{-1} \left[\frac{1 - \frac{C_{01}}{1+h}}{\sqrt{1 - C_{01}^2 C_{02}}} \right] + C_{05} \qquad (D.17)$$

Outer Expansion ϵ^1 Terms

The differential equations corresponding to the ϵ^1 equation of motion terms were derived in Section IV and are repeated below:

$$\frac{du_1}{dh} = -\frac{u_1}{1+h} \qquad (D.18)$$

$$\frac{dq_1}{dh} = \frac{q_0}{1+h} \left(\frac{2q_0 q_1}{u_0} - \frac{q_0^2 u_1}{u_0^2} \right) + \frac{q_1}{1+h} \left(\frac{q_0^2}{u_0} - 1 \right) \qquad (D.19)$$

$$\frac{dI_1}{dh} = 0 \qquad (D.20)$$

$$\frac{d\Omega_1}{dh} = 0 \qquad (D.21)$$

$$\frac{d\alpha_1}{dh} = -\frac{y_1}{(1+h)\sin^2(y_0)} \qquad (D.22)$$

du_1/dh Equation Solution. The Method of Separation of Variables solves Eq (D.18).

115

Solution: $u_1 = \dfrac{C_{11}}{1+h}$ (D.23)

<u>dq_1/dh Equation Solution</u>. Rearranging Eq (D.19) gives the ODE a form solved by using an integration factor.

$$\frac{dq_1}{dh} + \frac{q_1}{1+h}\left(1 - \frac{3q_0^2}{u_0}\right) = -\frac{q_0^3 u_1}{u_0^2(1+h)}$$

Substituting known relationships for u_0 (Eq (D.13)), q_0 (Eq (D.14)) and u_1 (Eq (D.23)) gives

$$\frac{dq_1}{dh} + \frac{q_1}{1+h}\left(1 - \frac{3}{2 - C_{01}C_{02}(1+h)}\right) = -\frac{C_{11}}{C_{01}^2\left(\frac{2}{C_{01}}(1+h) - C_{02}(1+h)^2\right)^{\frac{3}{2}}}$$

The ODE now has the form where an integrating factor is used to solve the ODE. From Eq(D.3)

$$P(h) = \frac{1}{1+h} - \frac{3}{(1+h)\left(2 - C_{01}C_{02}(1+h)\right)}$$

$$Q(h) = -\frac{C_{11}}{C_{01}^2\left(\frac{2}{C_{01}}(1+h) - C_{02}(1+h)^2\right)^{\frac{3}{2}}}$$

From Eq (D.3)

$$\upsilon = e^{\int P(h)\,dh} = e^{\int\left(\frac{1}{1+h} - \frac{3}{(1+h)\left(2 - C_{01}C_{02}(1+h)\right)}\right)dh}$$

$$= \frac{\left(2 - C_{01}C_{02}(1+h)\right)^{\frac{3}{2}}}{\sqrt{1+h}}$$

116

Thus, $Q(h)\upsilon - Q(h) e^{\int P(h) \, dh}$ gives

$$Q(h) e^{\int P(h) \, dh} = - \frac{C_{11}\left(\dfrac{\left(2 - C_{01}C_{02}(1 + h)\right)^{\frac{3}{2}}}{\sqrt{1 + h}}\right)}{C_{01}{}^2\left(\dfrac{2}{C_{01}}(1 + h) - C_{02}(1 + h)^2\right)^{\frac{3}{2}}}$$

$$= - \frac{C_{11}}{\sqrt{C_{01}}(1 + h)^2}$$

Integrating the above equation gives

$$\int Q(h) e^{\int P(h) \, dh} \, dh = \frac{C_{11}}{\sqrt{C_{01}}(1 + h)}$$

Thus, Eq (D.4) gives

$$\text{Solution: } q_1 = \frac{\left(\dfrac{C_{11}}{\sqrt{C_{01}}(1 + h)} + C_{12}\sqrt{1 + h}\right)}{\left(2 - C_{01}C_{02}(1 + h)\right)^{\frac{3}{2}}} \tag{D.24}$$

dI_1/dh Equation Solution. The solution to this ODE, Eq (D.20), is trivial.

$$\text{Solution: } I_1 = C_{13} \tag{D.25}$$

$d\Omega_1/dh$ Equation Solution. The solution to this ODE, Eq (D.21), is trivial.

$$\text{Solution: } \Omega_1 = C_{14} \tag{D.26}$$

117

<u>$d\alpha_1/dh$ Equation Solution</u>. This ODE is solved by first rewriting the term y_1 in Eq (D.22) in terms of q_1. From the assumed expansions for the outer variables (Eq (4.6)) and using the definition of $q = \cos(y)$

$$q = q_0 + q_1 + O(\epsilon^2) = \cos(y) = \cos(y_0 + y_1 + O(\epsilon^2))$$

The small perturbation expansion for the cosine function, Eq (C.8), gives

$$\cos(y_0 + y_1 + O(\epsilon^2)) = \cos(y_0) - y_1\sin(y_0)\,\epsilon + O(\epsilon^2)$$

Equating the above two expressions gives

$$q_0 = \cos(y_0) \tag{D.27}$$

$$y_1 = -\frac{q_1}{\sin(y_0)} \tag{D.28}$$

Therefore, Eq (D.22) becomes

$$\frac{d\alpha_1}{dh} = \frac{q_1}{(1 + h)\,\sin^3(y_0)}$$

Next, the $\sin^3(y_0)$ term in the above equation is expressed in terms of q_0, where, by definition, $q_0 = \cos(y_0)$.

$$\frac{1}{\sin^3(y_0)} = \frac{1}{(1 - \cos^2(y_0))^{\frac{3}{2}}} = \frac{1}{(1 - q_0^2)^{\frac{3}{2}}}$$

But from Eq (D.14)

$$q_0^2 = \frac{1}{\frac{2}{C_{01}}(1 + h) - C_{02}(1 + h)^2}$$

118

After combining the above two equations and simplifying, the expression for $1/\sin^3(\gamma_0)$ becomes

$$\frac{1}{\sin^3(\gamma_0)} = \left(\frac{2(1+h) - C_{01}C_{02}(1+h)^2}{-C_{01} + 2(1+h) - C_{01}C_{02}(1+h)^2}\right)^{\frac{3}{2}}$$

Substituting this relation into the ODE, Eq (D.22) gives

$$\frac{d\alpha_1}{dh} = \frac{q_1}{(1+h)}\left(\frac{2(1+h) - C_{01}C_{02}(1+h)^2}{-C_{01} + 2(1+h) - C_{01}C_{02}(1+h)^2}\right)^{\frac{3}{2}}$$

But from Eq (D.24), q_1 is known. Therefore, after substitution, the above ODE becomes

$$\frac{d\alpha_1}{dh} = \frac{1}{(1+h)}\left(\frac{2(1+h) - C_{01}C_{02}(1+h)^2}{-C_{01} + 2(1+h) - C_{01}C_{02}(1+h)^2}\right)^{\frac{3}{2}}\left(\frac{\frac{C_{11}}{\sqrt{C_{01}(1+h)}} + C_{12}\sqrt{1+h}}{\left(2 - C_{01}C_{02}(1+h)\right)^{\frac{3}{2}}}\right)$$

After simplifying, the ODE finally becomes

$$\frac{d\alpha_1}{dh} = \frac{\frac{C_{11}}{\sqrt{C_{01}}} + C_{12}(1+h)}{\left(-C_{01} + 2(1+h) - C_{01}C_{02}(1+h)^2\right)^{\frac{3}{2}}}$$

The Method of Separation of Variables is used to solve this ODE. From Eq (D.1), the solution has the form

$$\int d\alpha_1 = \alpha_1 = C_{15} + \frac{C_{11}}{\sqrt{C_{01}}} \int \frac{dh}{\left(-C_{01} + 2(1+h) - C_{01}C_{02}(1+h)^2\right)^{\frac{3}{2}}}$$

$$+ C_{12} \int \frac{(1+h)\,dh}{\left(-C_{01} + 2(1+h) - C_{01}C_{02}(1+h)^2\right)^{\frac{3}{2}}}$$

The first integral has the following form and solution (Beyer, 1984:255),.where $x = 1 + h$, $dx = dh$, $a = -C_{01}$, $b = 2$ and $c = -C_{01}C_{02}$.

$$\int \frac{dx}{\left(a + bx + cx^2\right)^{\frac{3}{2}}} = \frac{2(2cx + b)}{(4ac - b^2)\sqrt{a + bx + cx^2}} \quad \text{or}$$

$$\int \frac{dh}{\left(-C_{01} + 2(1+h) - C_{01}C_{02}(1+h)^2\right)^{\frac{3}{2}}}$$

$$= - \frac{-C_{01}C_{02}(1+h) + 1}{\left(1 - C_{01}^2C_{02}\right)\sqrt{-C_{01} + 2(1+h) - C_{01}C_{02}(1+h)^2}}$$

The second integral has the following form and solution (Beyer, 1984:256),.where $x = 1 + h$, $dx = dh$, $a = -C_{01}$, $b = 2$ and $c = -C_{01}C_{02}$.

$$\int \frac{x\,dx}{\left(a + bx + cx^2\right)^{\frac{3}{2}}} = - \frac{2(bx + 2a)}{(4ac - b^2)\sqrt{a + bx + cx^2}} + C \quad \text{or}$$

120

$$\int \frac{(1+h)\,dh}{\left(-C_{01} + 2(1+h) - C_{01}C_{02}(1+h)^2\right)^{\frac{3}{2}}}$$

$$- \frac{(1+h) - C_{01}}{\left(1 - C_{01}{}^2 C_{02}\right)\sqrt{-C_{01} + 2(1+h) - C_{01}C_{02}(1+h)^2}}$$

Substituting the above two integrations into the previous expression for α_1 gives

$$\alpha_1 = -\frac{C_{11}}{\sqrt{C_{01}}} \frac{-C_{01}C_{02}(1+h) + 1}{\left(1 - C_{01}{}^2 C_{02}\right)\sqrt{-C_{01} + 2(1+h) - C_{01}C_{02}(1+h)^2}}$$

$$+ \frac{C_{12}\left((1+h) - C_{01}\right)}{\left(1 - C_{01}{}^2 C_{02}\right)\sqrt{-C_{01} + 2(1+h) - C_{01}C_{02}(1+h)^2}} + C_{15}$$

Simplifying the above equation gives

$$\text{Solution: } \alpha_1 = \frac{\left(C_{12} + C_{11}C_{02}\sqrt{C_{01}}\right)(1+h) - \left(\frac{C_{11}}{\sqrt{C_{01}}} + C_{01}C_{12}\right)}{\left(1 - C_{01}{}^2 C_{02}\right)\sqrt{-C_{01} + 2(1+h) - C_{01}C_{02}(1+h)^2}} + C_{15} \quad (D.29)$$

Appendix E:
Derivation of Solutions for Inner Expansions of the Equations of Motion

Methods of Solution

The solutions to the two sets of five, coupled, first order, linear nonhomogeneous ordinary differential equations (ODEs) found in this appendix are derived below. Five methods are used to solve these equations. The first method used is the Method of Separation of Variables, where the ODE and its solution are given as (Beyer, 1984:315)

$$\frac{dy}{dx} = \frac{f(x)}{f(y)} \rightarrow \int f(y)\, dy + \int f(x)\, dx = K \tag{E.1}$$

K is the constant of integration.

The second method used to solve ODEs found in this appendix is the Method of Integrating Factors, where the ODE has the form

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{E.2}$$

The integrating factor has the form

$$\upsilon = e^{\int P(x)\, dx} \tag{E.3}$$

and the solution is given as (Beyer, 1984:315)

$$ye^{\int P(x)\, dx} = \int Q(x)\, e^{\int P(x)\, dx} + K \quad \text{or}$$

$$y = \frac{\int Q(x) e^{\int P(x)\,dx} + K}{e^{\int P(x)\,dx}} \tag{E.4}$$

The third method is a variation of the second method and is known as Bernoulli's Equation (Beyer, 1984:315). The ODE is given as

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{E.5}$$

Again, the integrating factor has the form

$$\upsilon = e^{\int P(x)\,dx} \tag{E.6}$$

and the solution is given, for n = 1, as (Beyer, 1984:315)

$$y^{(1-n)} e^{(1-n)\int P(x)\,dx} = (1-n)\int Q(x) e^{(1-n)\int P(x)\,dx} + K \quad \text{or}$$

$$y = \left(\frac{(1-n)\int Q(x) e^{(1-n)\int P(x)\,dx} + K}{e^{(1-n)\int P(x)\,dx}}\right)^{\frac{1}{1-n}} \tag{E.7}$$

The fourth method occurs when the ODE does not have an elementary function as its solution. To derive a solution to the ODE, judicious approximations are made, using binomial and Taylor expansions and assumptions based on the physics of atmospheric entry and on experience.

The fifth and final method to solve the following ODEs recognizes that some of the ODEs are coupled and cannot be solved independently. Thus, an intermediate solution is found to two (or more) ODEs and by using this intermediate solution, the original ODEs are solved (Karasopoulos,

1988:177). For example, suppose there are two ODEs coupled and cannot be solved independently, $dy/dx = f(x,y,z)$ and $dz/dx = f(x,y,z)$. If the two ODEs are combined by a mathematical operation, like addition or division, the independent variable, x, may possibly disappear, resulting in an ODE, $dz/dy = f(y,z)$, which is solvable. Assuming there is a solution to the above ODE in the form $z = F1(y,z)$ or $y = F2(y,z)$, the two original ODEs are expressed as functions of only two variables and thus are solvable.

Constants of Integration Notation

As shown in Section IV, there are ten ODEs expanded from the equations of motion. Accompanying the solution of each of these ODEs is a constant of integration. To simplify the bookkeeping involved in defining all these constants, the following notation is used for the outer expansion solutions:

The constant of integration is given as K_{ij}, where i denotes the order $(\epsilon^i -- \epsilon^0$ or $\epsilon^1)$ of the solution and j denotes the variable associated with this constant.

j	variable
1	u
2	q
3	I
4	Ω
5	α

Thus K_{01} is the constant of integration for the variable u_0 and K_{14} is the constant integration for the variable Ω_1. In addition, the coupling present in the first order inner ODEs generates lengthy constant terms in their solutions. The following shorthand notation is used to describe these

124

terms. K_{1jk} are the constants multiplying the terms in the integral solution. As before, 'K' signifies an inner expansion solution, '1' signifies a first order solution, 'j' signifies the variable being solved and 'k' signifies the kth constant being defined in this solution. K_{141} is the first multiplicative constant for the solution of variable Ω_1. This shorthand will only be used for the first order inner solutions.

Inner Expansion ϵ^0 Terms

The differential equations corresponding to the ϵ^0 equation of motion terms were derived in Section IV and are repeated below:

$$\frac{du_0}{d\xi} = \frac{-2Bu_0e^{-\xi}\left(1 + \lambda\tan(\gamma_0)\right)}{\sin(\gamma_0)} \tag{E.8}$$

$$\frac{dq_0}{d\xi} = -\lambda Be^{-\xi} \tag{E.9}$$

$$\frac{dI_0}{d\xi} = \frac{B\delta e^{-\xi}\cos(\alpha_0)}{\sin(\gamma_0)\cos(\gamma_0)} \tag{E.10}$$

$$\frac{d\Omega_0}{d\xi} = \frac{B\delta e^{-\xi}\sin(\alpha_0)}{\sin(I_0)\sin(\gamma_0)\cos(\gamma_0)} \tag{E.11}$$

$$\frac{d\alpha_0}{d\xi} = -\frac{B\delta e^{-\xi}\sin(\alpha_0)}{\tan(I_0)\sin(\gamma_0)\cos(\gamma_0)} \tag{E.12}$$

$dq_0/d\xi$ Equation Solution. The Method of Separation of Variables solves Eq (E.9). Rearranging Eq (E.9) into form of Eq (E.1) gives

$$\frac{dq_0}{d\xi} = -\lambda Be^{-\xi} \Rightarrow dq_0 = -\lambda Be^{-\xi}d\xi \Rightarrow \int dq_0 = -\int \lambda Be^{-\xi}d\xi + K_{02}$$

125

Solving the above ODE gives

$$\text{Solution: } q_0 = \lambda B e^{-\xi} + K_{02} \tag{E.13}$$

$du_0/d\xi$ Equation Solution. This ODE is solved by first rewriting the terms $\sin(\gamma_0)$ and $\cos(\gamma_0)$ in Eq (E.8) in terms of q_0, where $q_0 = \cos(\gamma_0)$. Thus, $\cos(\gamma_0) = q_0$ and $\sin(\gamma_0) = \sqrt{1-\cos^2(\gamma_0)} = \sqrt{1-q_0^2}$. Substituting these into the ODE gives

$$\frac{du_0}{d\xi} = -2Bu_0 e^{-\xi}\left(\frac{1}{\sin(\gamma_0)} + \frac{\lambda}{\cos(\gamma_0)}\right) = -2Bu_0 e^{-\xi}\left(\frac{1}{\sqrt{1-q_0^2}} + \frac{\lambda}{q_0}\right)$$

As derived in Eq (E.13), $q_0 = q_0(\xi)$. To simplify the above integration, the independent variable is changed from ξ to q_0. Since $q_0 = \lambda B e^{-\xi} + K_{02}$, $dq_0 = -\lambda B e^{-\xi} d\xi$. Thus, the ODE now becomes

$$\frac{du_0}{dq_0} = \frac{2u_0}{\lambda}\left(\frac{1}{\sqrt{1-q_0^2}} + \frac{\lambda}{q_0}\right)$$

Now, the Method of Separation of Variables is used to solve the above equation. Rearranging the above equation into form of Eq (E.1) gives

$$\frac{du_0}{dq_0} = \frac{2u_0}{\lambda}\left(\frac{1}{\sqrt{1-q_0^2}} + \frac{\lambda}{q_0}\right) \Rightarrow \frac{du_0}{u_0} = \frac{2}{\lambda}\left(\frac{1}{\sqrt{1-q_0^2}} + \frac{\lambda}{q_0}\right)dq_0$$

$$\Rightarrow \int \frac{du_0}{u_0} = \int \frac{2}{\lambda}\left(\frac{1}{\sqrt{1-q_0^2}} + \frac{\lambda}{q_0}\right)dq_0 + \tilde{K}_{01}$$

Solving the above integral (Beyer, 1984:252) and remembering, by definition, (Eq (D.27)) $q_0 = \cos(\gamma_0)$ or $\gamma_0 = \text{Cos}^{-1}(q_0)$

126

$$\ln\left(u_0\right) = -\frac{2}{\lambda}\, \text{Cos}^{-1}\left(q_0\right) + 2\,\ln\left(q_0\right) + \tilde{K}_{01}$$

$$\ln\left(u_0\right) = -\frac{2\gamma_0}{\lambda} + \ln\left(q_0{}^2\right) + \tilde{K}_{01}$$

Rearranging the above equation gives the final solution, where $\exp\left(\tilde{K}_{01}\right) = K_{01}$

$$\text{Solution: } u_0 = K_{01}\, q_0{}^2\, \exp\left[-\frac{2}{\lambda}\, \text{Cos}^{-1}(q_0)\right]$$

$$= K_{01}\, q_0{}^2\, \exp\left[-\frac{2}{\lambda}\, \gamma_0\right]$$

$$= K_{01}\left(\lambda B e^{-\xi} + K_{02}\right)^2 \exp\left[-\frac{2}{\lambda}\, \text{Cos}^{-1}\left(\lambda B e^{-\xi} + K_{02}\right)\right] \quad (E.14)$$

$dI_0/d\xi$ Equation Solution. The last three ODEs are coupled and are solved as such. First, the I_0 and α_0 ODEs are combined to express I_0 as a function of α_0. Using the chain rule for differentiation

$$\frac{dI_0}{d\alpha_0} = \frac{dI_0}{d\xi}\,\frac{d\xi}{d\alpha_0}$$

Substituting for the two known ODEs (Eqs (E.10) and (E.12)) on the right hand side

$$\frac{dI_0}{d\alpha_0} = -\frac{B\delta e^{-\xi}\cos(\alpha_0)}{\sin(\gamma_0)\cos(\gamma_0)}\,\frac{\tan(I_0)\sin(\gamma_0)\cos(\gamma_0)}{B\delta e^{-\xi}\sin(\alpha_0)} = -\frac{\tan(I_0)}{\tan(\alpha_0)}$$

Separation of variables solves the above equation. Rearranging the above equation into form of Eq (E.1) gives I_0 as a function of α_0 (Beyer, 1984:260).

127

$$\frac{dI_0}{d\alpha_0} = -\frac{\tan(I_0)}{\tan(\alpha_0)} \implies \frac{dI_0}{\tan(I_0)} = -\frac{d\alpha_0}{\tan(\alpha_0)}$$

$$\implies \int \frac{dI_0}{\tan(I_0)} + \int \frac{d\alpha_0}{\tan(\alpha_0)} = \ln\left(\sin(K_{03})\right)$$

$$\implies \ln\left(\sin(I_0)\right) + \ln\left(\sin(\alpha_0)\right) = \ln\left(\sin(K_{03})\right)$$

$$\implies \sin(I_0)\sin(\alpha_0) = \sin(K_{03}) \quad \text{or}$$

$$I_0 = Sin^{-1}\left(\frac{\sin(K_{03})}{\sin(\alpha_0)}\right) \quad \text{and} \quad \alpha_0 = Sin^{-1}\left(\frac{\sin(K_{03})}{\sin(I_0)}\right) \qquad (E.15)$$

Next, the Ω_0 and α_0 ODEs are combined. Using the chain rule for differentiation

$$\frac{d\alpha_0}{d\Omega_0} = \frac{d\alpha_0}{d\xi}\frac{d\xi}{d\Omega_0}$$

Substituting for the two known ODEs (Eqs (E.11) and (E.12)) on the right hand side

$$\frac{d\alpha_0}{d\Omega_0} = -\frac{B\delta e^{-\xi}\sin(\alpha_0)}{\tan(I_0)\sin(\gamma_0)\cos(\gamma_0)}\frac{\sin(I_0)\sin(\gamma_0)\cos(\gamma_0)}{B\delta e^{-\xi}\sin(\alpha_0)} = -\cos(I_0)$$

But from Eq (E.15), I_0 is known. Using this and the trigonometric identity $\cos\left(Sin^{-1}(x)\right) = \sqrt{1 - x^2}$ (Beyer, 1984:141), the above expression is transformed as follows.

$$\frac{d\alpha_0}{d\Omega_0} = -\cos(I_0) = -\cos\left[Sin^{-1}\left(\frac{\sin(K_{03})}{\sin(\alpha_0)}\right)\right]$$

128

$$= - \left[1 - \left(\frac{\sin(K_{03})}{\sin(\alpha_0)} \right)^2 \right]^{\frac{1}{2}}$$

$$= - \left[\frac{\sin^2(\alpha_0) - \sin^2(K_{03})}{\sin^2(\alpha_0)} \right]^{\frac{1}{2}}$$

Separation of variables solves the above equation. Rearranging the above equation into form of Eq (E.1) gives

$$\frac{d\alpha_0}{d\Omega_0} = - \left[\frac{\sin^2(\alpha_0) - \sin^2(K_{03})}{\sin^2(\alpha_0)} \right]^{\frac{1}{2}} \Rightarrow d\Omega_0 = - \frac{\sin(\alpha_0)\, d\alpha_0}{\sqrt{\sin^2(\alpha_0) - \sin^2(K_{03})}}$$

$$\Rightarrow \Omega_0 = - \int \frac{\sin(\alpha_0)\, d\alpha_0}{\sqrt{\sin^2(\alpha_0) - \sin^2(K_{03})}} + K_{04}$$

Using the substitutions $x = \cos(\alpha_0)$, $dx = - \sin(\alpha_0)\, d\alpha_0$ and $\sin^2(\alpha_0) = 1 - \cos^2(\alpha_0) = 1 - x^2$, the above integral is easily solved (Beyer, 1984:252) and gives Ω_0 as a function of α_0.

$$\Omega_0 = - \int \frac{\sin(\alpha_0)\, d\alpha_0}{\sqrt{\sin^2(\alpha_0) - \sin^2(K_{03})}} + K_{04}$$

$$= - \int \frac{dx}{\sqrt{1 - x^2 - \sin^2(K_{03})}} + K_{04}$$

$$= - \int \frac{dx}{\sqrt{\cos^2(K_{03}) - x^2}} + K_{04}$$

$$= - \mathrm{Cos}\left(\frac{\cos(\alpha_0)}{\cos(K_{03})} \right) + K_{04} \quad \text{or}$$

$$\cos\left(K_{04} - \Omega_0\right) = \frac{\cos(\alpha_0)}{\cos(K_{03})} \qquad (E.16)$$

Finally, the $dI_0/d\xi$ ODE is transformed to $dI_0/d\gamma_0$ via the chain rule for differentiation, use of Eq (E.9) and the definition $q_0 = \cos(\gamma_0)$

$$-\lambda B e^{-\xi} = \frac{dq_0}{d\xi} = \frac{d\left(\cos(\gamma_0)\right)}{d\xi} = -\sin(\gamma_0)\frac{d\gamma_0}{d\xi} \Rightarrow \frac{d\gamma_0}{d\xi} = \frac{\lambda B e^{-\xi}}{\sin(\gamma_0)}$$

Using this relation and Eq (E.10) gives

$$\frac{dI_0}{d\gamma_0} = \frac{dI_0}{d\xi}\frac{d\xi}{d\gamma_0} = \frac{B\delta e^{-\xi}\cos(\alpha_0)}{\sin(\gamma_0)\cos(\gamma_0)}\frac{\sin(\gamma_0)}{\lambda B e^{-\xi}} = \frac{\delta\cos(\alpha_0)}{\lambda\cos(\gamma_0)}$$

Substituting the relation for $\sin(\alpha_0)$ in Eq (E.15) gives

$$\frac{dI_0}{d\gamma_0} = \frac{\delta\left[1 - \dfrac{\sin^2(K_{03})}{\sin^2(I_0)}\right]^{1/2}}{\lambda\cos(\gamma_0)} = \frac{\delta\left[\sin^2(I_0) - \sin^2(K_{03})\right]^{1/2}}{\lambda\,\sin(I_0)\cos(\gamma_0)}$$

Separation of variables solves the above equation. Rearranging the above equation into form of Eq (E.1) gives

$$\frac{\delta}{\lambda}\sec(\gamma_0)\,d\gamma_0 = \frac{\sin(I_0)\,dI_0}{\sqrt{\sin^2(I_0) - \sin^2(K_{03})}}\quad \text{or}$$

$$\frac{\delta}{\lambda}\int\sec(\gamma_0)\,d\gamma_0 = \int\frac{\sin(I_0)\,dI_0}{\sqrt{\sin^2(I_0) - \sin^2(K_{03})}} + K_{05}$$

The first integral is easily solved (Beyer, 1984:260)

$$\frac{\delta}{\lambda}\int\sec(\gamma_0)\,d\gamma_0 = \frac{\delta}{\lambda}\ln\left[\tan\left(\frac{\pi}{4} + \frac{\gamma_0}{2}\right)\right]$$

Using the substitutions $x = \cos(I_0)$, $dx = -\sin(I_0)\,dI_0$ and $\sin^2(I_0) = 1 - \cos^2(I_0) = 1 - x^2$, the second integral is easily solved (Beyer, 1984:252) and gives

$$\frac{\delta}{\lambda}\ln\left[\tan\left(\frac{\pi}{4} + \frac{\gamma_0}{2}\right)\right] = \int \frac{\sin(I_0)\,dI_0}{\sqrt{\sin^2(I_0) - \sin^2(K_{03})}} - K_{05}$$

$$= -\int \frac{dx}{\sqrt{1 - x^2 - \sin^2(K_{03})}} - K_{05}$$

$$= -\int \frac{dx}{\sqrt{\cos^2(K_{03}) - x^2}} - K_{05}$$

$$= \text{Cos}^{-1}\left(\frac{\cos(I_0)}{\cos(K_{03})}\right) - K_{05} \quad \text{or}$$

$$\cos(I_0) = \cos(K_{03})\cos\left(\frac{\delta}{\lambda}\ln\left[\tan\left(\frac{\pi}{4} + \frac{\gamma_0}{2}\right)\right] + K_{05}\right)$$

Solution: $I_0 = \text{Cos}^{-1}\left\{\cos(K_{03})\cos\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4} + \frac{\gamma_0}{2}\right)\right) + K_{05}\right]\right\}$

$$= \text{Cos}^{-1}\left\{\cos(K_{03})\cos\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4} + \frac{\text{Cos}^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)}{2}\right)\right) + K_{05}\right]\right\} \quad \text{(E.17)}$$

$d\alpha_0/d\xi$ Equation Solution. This solution is derived above and is repeated for continuity. From Eqs (E.15) and (E.17) α_0 is expressed in terms of the independent variable, ξ.

Solution: $\alpha_0 = \text{Sin}^{-1}\left(\frac{\sin(K_{03})}{\sin(I_0)}\right)$

$$- \sin^{-1}\left[\frac{\sin(K_{03})}{\left[1 - \cos^2(K_{03})\cos^2\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4}+\frac{\cos^{-1}\left(\lambda Be^{-\xi}+K_{02}\right)}{2}\right)\right)+K_{05}\right]\right]^{\frac{1}{2}}}\right] \quad (E.18)$$

$d\Omega_0/d\xi$ Equation Solution.

This solution is derived above and is repeated for continuity. From Eqs (E.16) and (E.17) Ω_0 is expressed in terms of the independent variable, ξ.

$$\text{Solution: } \Omega_0 - K_{04} - \cos^{-1}\left(\frac{\cos(\alpha_0)}{\cos(K_{03})}\right) - K_{04} \quad (E.19)$$

$$- \cos^{-1}\left(\frac{\left\{1 - \left[\frac{\sin^2(K_{03})}{1 - \cos^2(K_{03})\cos^2\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4}+\frac{\cos^{-1}\left(\lambda Be^{-\xi}+K_{02}\right)}{2}\right)\right)+K_{05}\right]}\right]^2\right\}^{\frac{1}{2}}}{\cos(K_{03})}\right)$$

Inner Expansion ϵ^1 Terms

The differential equations corresponding to the ϵ^1 equation of motion terms were derived in Section IV and are repeated below:

$$\frac{du_1}{d\xi} - -u_0 - 2Be^{-\xi}\left[\frac{u_1\left(1 + \lambda\tan(\gamma_0)\right)}{\sin(\gamma_0)}\right.$$

$$\left. - \frac{u_0\gamma_1\cos(\gamma_0)\left(1 + \lambda\tan(\gamma_0)\right)}{\sin^2(\gamma_0)} + \frac{\lambda\,u_0\,\gamma_1}{\cos^2(\gamma_0)\,\sin(\gamma_0)}\right] \quad (E.20)$$

$$\frac{dq_1}{d\xi} - q_0\left(-1 + \frac{q_0^2}{u_0}\right) \quad (E.21)$$

132

$$\frac{dI_1}{d\xi} = B\delta e^{-\xi}\left[\gamma_1\cos(\alpha_0)\left(\frac{1}{\cos^2(\gamma_0)} - \frac{1}{\sin^2(\gamma_0)}\right) - \frac{\alpha_1\sin(\alpha_0)}{\sin(\gamma_0)\cos(\gamma_0)}\right] \quad (E.22)$$

$$\frac{d\Omega_1}{d\xi} = B\delta e^{-\xi}\left[\frac{\gamma_1\sin(\alpha_0)}{\sin(I_0)}\left(\frac{1}{\cos^2(\gamma_0)} - \frac{1}{\sin^2(\gamma_0)}\right)\right.$$

$$\left. - \frac{I_1\sin(\alpha_0)\cos(I_0)}{\sin^2(I_0)\sin(\gamma_0)\cos(\gamma_0)} + \frac{\alpha_1\cos(\alpha_0)}{\sin(I_0)\sin(\gamma_0)\cos(\gamma_0)}\right] \quad (E.23)$$

$$\frac{d\alpha_1}{d\xi} = \frac{1}{\tan(\gamma_0)} - B\delta e^{-\xi}\left[\frac{\gamma_1\sin(\alpha_0)}{\tan(I_0)}\left(\frac{1}{\cos^2(\gamma_0)} - \frac{1}{\sin^2(\gamma_0)}\right)\right.$$

$$\left. - \frac{I_1\sin(\alpha_0)}{\sin^2(I_0)\sin(\gamma_0)\cos(\gamma_0)} + \frac{\alpha_1\cos(\alpha_0)}{\tan(I_0)\sin(\gamma_0)\cos(\gamma_0)}\right] \quad (E.24)$$

$dq_1/d\xi$ Equation Solution. The Method of Separation of Variables solves Eq (E.21). Substituting relations for u_0 (Eq (E.13)) and q_0 (Eq (E.14)) into Eq (E.21) and simplifying gives

$$\frac{dq_1}{d\xi} = q_0\left(-1 + \frac{q_0^2}{u_0}\right)$$

$$= \left(\lambda Be^{-\xi} + K_{02}\right)\left(-1 + \frac{\left(\lambda Be^{-\xi} + K_{02}\right)^2}{K_{01}\left(\lambda Be^{-\xi} + K_{02}\right)^2\exp\left[-\frac{2}{\lambda}\mathrm{Cos}^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)\right]}\right)$$

$$= \left(\lambda Be^{-\xi} + K_{02}\right)\left(\frac{\exp\left[\frac{2}{\lambda}\mathrm{Cos}^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)\right]}{K_{01}} - 1\right)$$

Rearranging the above equation into form of Eq (E.1) gives

133

$$\int dq_1 = \int \left(\lambda Be^{-\xi} + K_{02}\right)\left(\frac{\exp\left[\frac{2}{\lambda}\cos^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)\right]}{K_{01}} - 1\right)d\xi + K_{12}$$

$$q_1 = \lambda Be^{-\xi} - K_{02}\xi$$

$$+ \frac{1}{K_{01}}\int \left(\lambda Be^{-\xi} + K_{02}\right)\exp\left[\frac{2}{\lambda}\cos^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)\right]d\xi + K_{12}$$

Using the substitutions $x = \lambda Be^{-\xi} + K_{02}$, $x = \cos(\phi)$ and the trigonometric relationship $\frac{2\sin(\phi)}{\cos(d) - \cos(\phi)} = \cot\left(\frac{\phi - d}{2}\right) + \cot\left(\frac{\phi + d}{2}\right)$ (Karameheti, 1966:624), the above integral is reduced to

$$\int \left(\lambda Be^{-\xi} + K_{02}\right)\exp\left[\frac{2}{\lambda}\cos^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)\right]d\xi$$

$$= \int \frac{x}{K_{02} - x}\exp\left[-\frac{2}{\lambda}\cos^{-1}(x)\right]dx$$

$$= -\int \frac{\cos(\phi)\sin(\phi)\exp\left(\frac{2}{\lambda}\phi\right)}{\cos(d) - \cos(\phi)}d\phi \quad (\cos(d) = K_{02})$$

$$= -\frac{1}{2}\int \cos(\phi)\left\{\cot\left(\frac{\phi - d}{2}\right) + \cot\left(\frac{\phi + d}{2}\right)\right\}\exp\left(\frac{2}{\lambda}\phi\right)d\phi$$

The above integral can be reduced to a sum of simpler integrals, but one resulting integral has an integrand of $\cot(z)e^{bz}$, where z is the integration variable, a function of ϕ, and b is a constant. Using integration by parts, the resulting integral has an integrand of $e^{bz}/\sin^2 z$, whose integral cannot be expressed as a finite sum of elementary integrals (Gradshteyn

134

and Ryzhik, 1980:197). Thus, the approximation made to solve the above integrals is given as

$$\int \left(\lambda B e^{-\xi} + K_{02} \right) \exp\left[\frac{2}{\lambda} \text{Cos}^{-1} \left(\lambda B e^{-\xi} + K_{02} \right) \right] d\xi$$

$$\approx \int \left(\lambda B e^{-\xi} + K_{02} \right) \exp\left[\frac{2}{\lambda} \text{Cos}^{-1} \left(K_{02} \right) \right] d\xi$$

Figure E1 compares the relationship between the exact function (the integrand on the left hand side above) and the approximate function. While there is a significant disparity at small values of ξ (low orbital altitudes), the above difference is negligible when both functions are integrated from ξ_0 (orbital altitude at the beginning of atmospheric entry) to $\xi \approx 0$. Figure E2 shows the integrals of the exact and approximate functions are identical until $\xi \to 0$, where the two functions deviate slightly. This trend is reflected in Figure E.3, which compares the exact solution of q_1, obtained from numerical integration of q_1, and the approximate function derived for q_1. Thus substituting the approximate function into the above solution for q_1 gives an approximate solution for q_1.

Solution: $\quad q_1 \approx K_{121} \left(K_{02} \xi - \lambda B e^{-\xi} \right) + K_{12}$ (E.25)

where $\quad K_{121} \approx \dfrac{\exp\left(\dfrac{2}{\lambda} \text{Cos}^{-1}(K_{02}) \right)}{K_{01}} - 1$ (E.26)

$\underline{du_1/d\xi \ \text{Equation Solution}}$. Rearranging Eq (E.20) gives the ODE a form solved by using an integration factor.
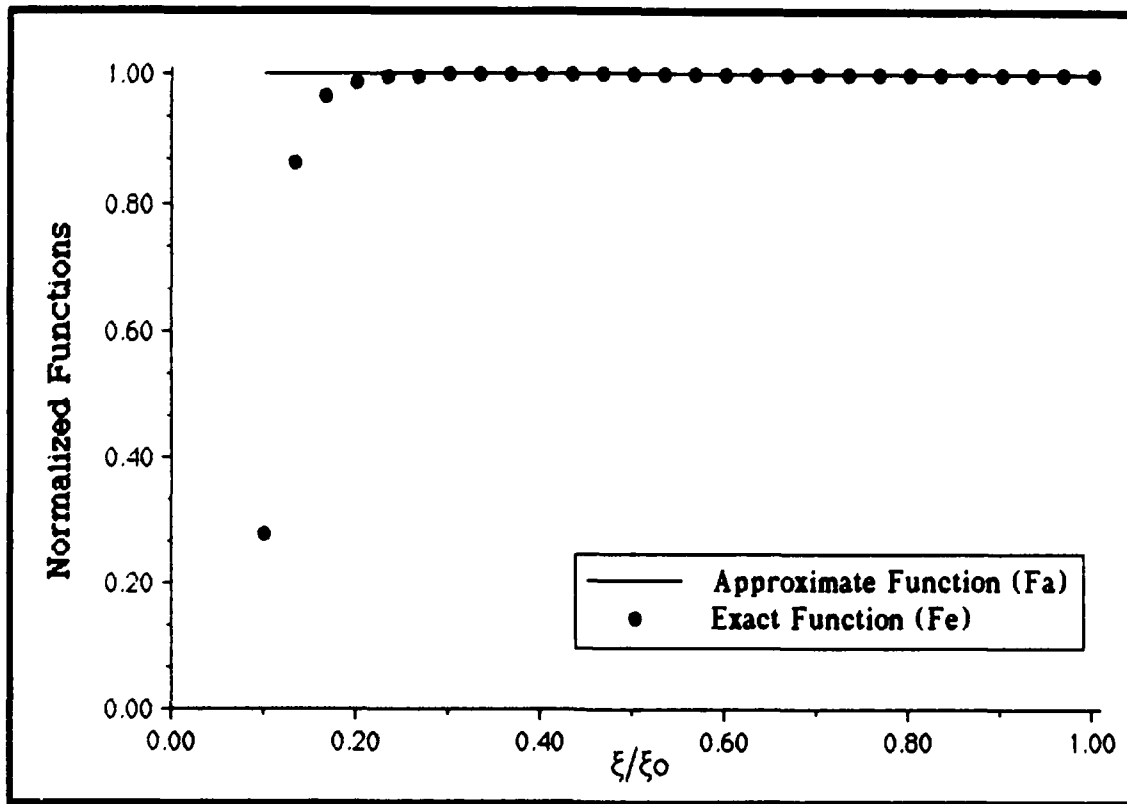
135

Figure E1.   Comparison of Approximate and Exact Functions
in Deriving the q_1 Solution

$$\frac{du_1}{d\xi} + \left(\frac{2B\left(1 + \lambda\tan(\gamma_0)\right)e^{-\xi}}{\sin(\gamma_0)}\right)u_1 = -u_0 + \frac{2Bu_0\gamma_1\cos(\gamma_0)\left(1 + \lambda\tan(\gamma_0)\right)e^{-\xi}}{\sin^2(\gamma_0)}$$

$$- \frac{2B\lambda\, u_0\, \gamma_1 e^{-\xi}}{\cos^2(\gamma_0)\sin(\gamma_0)}$$

The ODE now has the form where an integrating factor is used to solve the ODE.

$$P(\xi) = \frac{2B\left(1 + \lambda\tan(\gamma_0)\right)e^{-\xi}}{\sin(\gamma_0)}$$

136
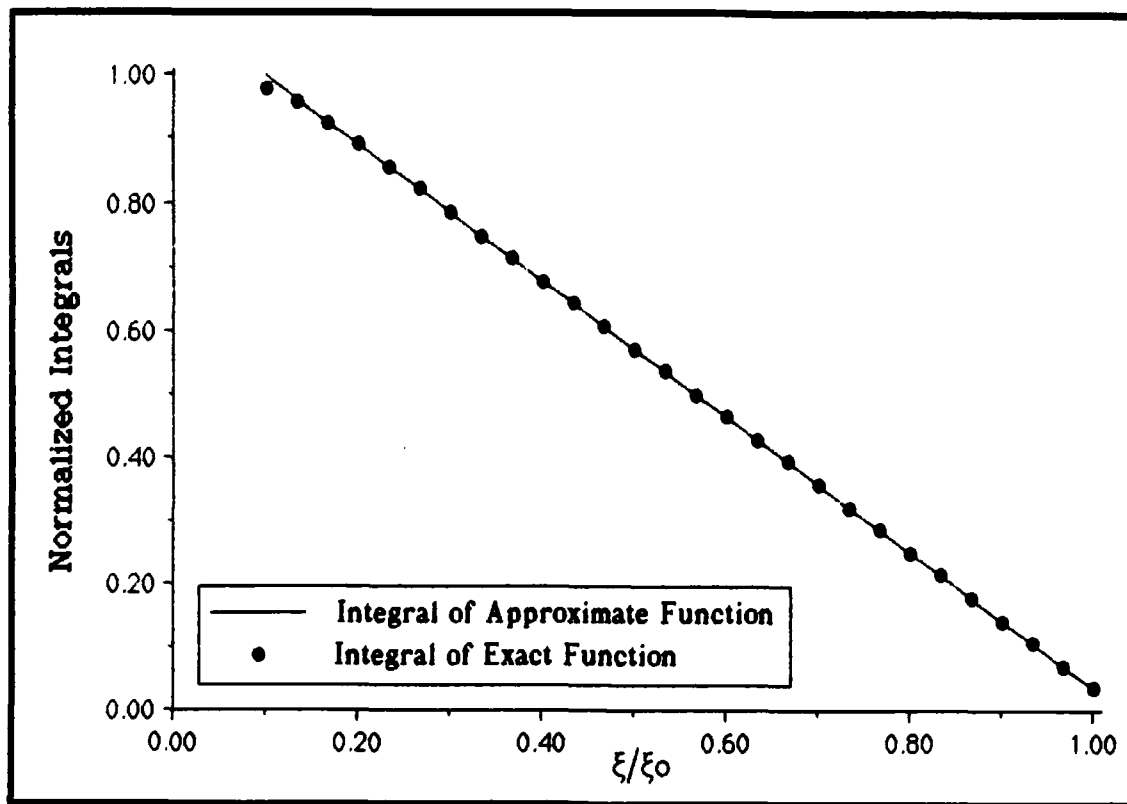
Figure E2.   Comparison of the Integrations of the Approximate
and Exact Functions in Deriving the q_1 Solution

$$Q(\xi) = -u_0 + \frac{2Bu_0 \gamma_1 \cos(\gamma_0)\left(1 + \lambda\tan(\gamma_0)\right)e^{-\xi}}{\sin^2(\gamma_0)} - \frac{2B\lambda\, u_0\, \gamma_1 e^{-\xi}}{\cos^2(\gamma_0)\,\sin(\gamma_0)}$$

From Eq (E.3)

$$\upsilon = e^{\int P(\xi)\, d\xi} = \exp\int \frac{2B\left(1 + \lambda\tan(\gamma_0)\right)e^{-\xi}}{\sin(\gamma_0)}\, d\xi$$

Factoring the trigonometric terms and noting $\cos(\gamma_0) = q_0$ and $\sin(\gamma_0)$
$= \sqrt{1 - q_0^2}$, the above integral in separated into the sum of two
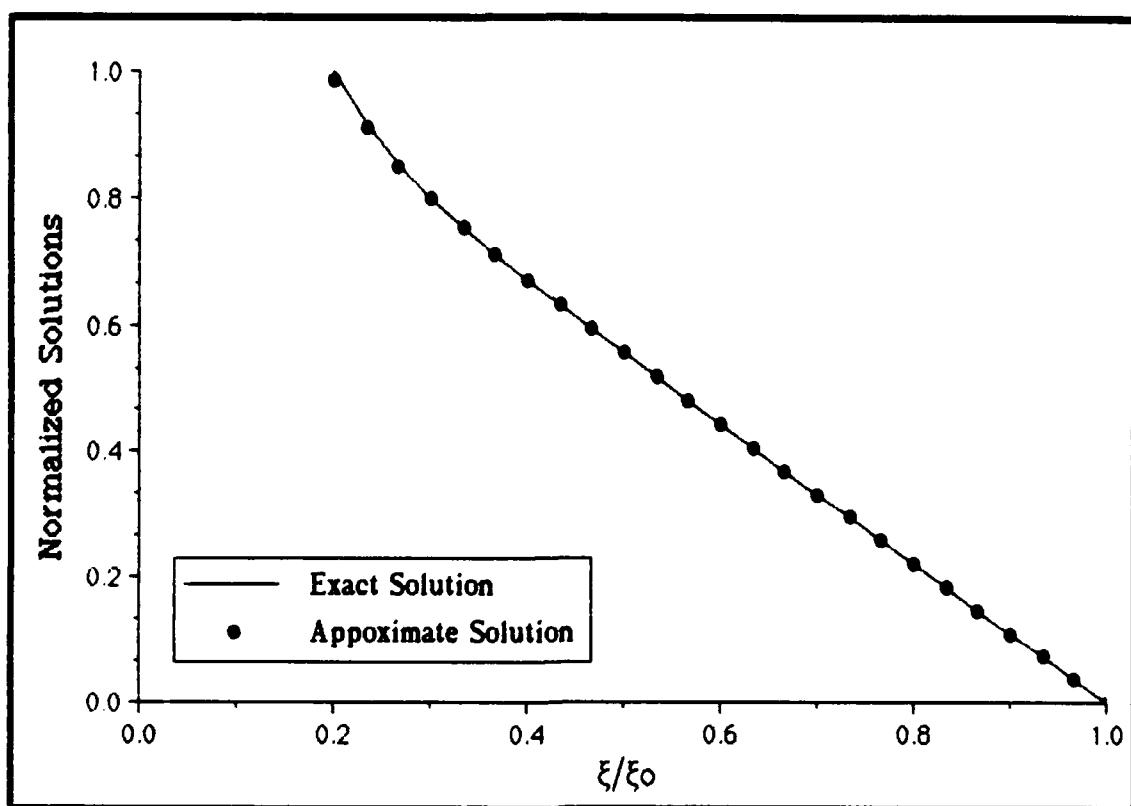
137

Figure E3. Comparison of the Approximate and Exact Integral
Solutions for q_1

integrals. Substituting in the derived expression $q_0 = \lambda Be^{-\xi} + K_{02}$

(Eq (E.13)), gives

$$v = \exp \int \left(\frac{2Be^{-\xi}}{\sqrt{1 - \left(\lambda Be^{-\xi} + K_{02} \right)^2}} + \frac{2B\lambda e^{-\xi}}{\lambda Be^{-\xi} + K_{02}} \right) d\xi$$

Using the substitution $x = \lambda Be^{-\xi} + K_{02}$, the above integral is easily solved.

$$v = \exp \left[\frac{2}{\lambda} \cos^{-1} \left(\lambda Be^{-\xi} + K_{02} \right) - 2 \ln \left(\lambda Be^{-\xi} + K_{02} \right) \right]$$

138

$$= \frac{\exp\left(\frac{2}{\lambda}\text{Cos}^{-1}\left(\lambda B e^{-\xi} + K_{02}\right)\right)}{\left(\lambda B e^{-\xi} + K_{02}\right)^2}$$

$$= \frac{\exp\left(\frac{2\gamma_0}{\lambda}\right)}{q_0^2}$$

Thus, after factoring the trigonometric terms and using the definition for q_0 as above, and substituting the expression for q_1 (Eq (E.25)), where by definition (Eq (D.27)) $\gamma_1 = -q_1/\sin(\gamma_0)$, $Q(\xi)\upsilon = Q(\xi)\, e^{\int P(\xi)\, d\xi}$ gives

$$Q(\xi)\upsilon = \frac{\exp\left(\frac{2\gamma_0}{\lambda}\right)}{q_0^2}\left[-u_0 + \frac{2Bu_0\gamma_1\cos(\gamma_0)\left(1+\lambda\tan(\gamma_0)\right)e^{-\xi}}{\sin^2(\gamma_0)} - \frac{2B\lambda\, u_0\,\gamma_1 e^{-\xi}}{\cos^2(\gamma_0)\,\sin(\gamma_0)}\right]$$

$$= - K_{01} + \frac{2B\lambda K_{01}e^{-\xi}}{\left(\lambda B e^{-\xi} + K_{02}\right)^2}\left[-K_{121}\lambda B e^{-\xi} + K_{02}\, K_{121}\xi + K_{12}\right]$$

$$- \frac{2BK_{01}e^{-\xi}\left(\lambda B e^{-\xi} + K_{02}\right)}{\left[1 - \left(\lambda B e^{-\xi} + K_{02}\right)^2\right]^{\frac{3}{2}}}\left[-K_{121}\lambda B e^{-\xi} + K_{02}\, K_{121}\xi + K_{12}\right]$$

Distributing the terms in the above equation and integrating gives (where $A_1 - A_6$ are constants)

$$\int Q(\xi)\upsilon\, d\xi = K_{11} - K_{01}\xi + A_1\int \frac{\lambda B e^{-\xi}}{\left(\lambda B e^{-\xi} + K_{02}\right)^2}\, d\xi$$

139

$$+ A_2 \int \frac{\lambda^2 B^2 e^{-2\xi}}{\left(\lambda Be^{-\xi} + K_{02}\right)^2} \, d\xi + A_3 \int \frac{\lambda Be^{-\xi} \xi}{\left(\lambda Be^{-\xi} + K_{02}\right)^2} \, d\xi$$

$$+ A_4 \int \frac{\lambda Be^{-\xi} \left(\lambda Be^{-\xi} + K_{02}\right)}{\left[1 - \left(\lambda Be^{-\xi} + K_{02}\right)^2\right]^{\frac{3}{2}}} \, d\xi$$

$$+ A_5 \int \frac{\lambda^2 B^2 e^{-2\xi} \left(\lambda Be^{-\xi} + K_{02}\right)}{\left[1 - \left(\lambda Be^{-\xi} + K_{02}\right)^2\right]^{\frac{3}{2}}} \, d\xi$$

$$+ A_6 \int \frac{\lambda Be^{-\xi} \left(\lambda Be^{-\xi} + K_{02}\right) \xi}{\left[1 - \left(\lambda Be^{-\xi} + K_{02}\right)^2\right]^{\frac{3}{2}}} \, d\xi$$

The first and second integrals above are easily solved by using the substitution $x = \lambda Be^{-\xi} + K_{02}$. The third integral is solved by using the following relation (integration by parts and (Beyer, 1984:278)).

$$\int \frac{\lambda Be^{-\xi} \xi}{\left(\lambda Be^{-\xi} + K_{02}\right)^2} \, d\xi = \frac{\xi}{\lambda Be^{-\xi} + K_{02}} - \int \frac{d\xi}{\lambda Be^{-\xi} + K_{02}}$$

$$= \frac{\left(1 - \frac{1}{K_{02}}\right)\xi}{\lambda Be^{-\xi} + K_{02}} - \frac{1}{K_{02}} \ln\left(\lambda Be^{-\xi} + K_{02}\right)$$

The fourth and fifth integrals are easily solved using the substitution $x = \lambda Be^{-\xi} + K_{02}$ and the following relation (Beyer, 1984:253,254).

140

$$\int \frac{\lambda B e^{-\xi}\left(\lambda B e^{-\xi} + K_{02}\right)}{\left[1 - \left(\lambda B e^{-\xi} + K_{02}\right)^2\right]^{\frac{3}{2}}} \, d\xi = \int \frac{x^2 \, dx}{\left[1 - x^2\right]^{\frac{3}{2}}} - \int \frac{x \, dx}{\left[1 - x^2\right]^{\frac{3}{2}}}$$

$$= \frac{\lambda B e^{-\xi}}{\sqrt{1 - \left(\lambda B e^{-\xi} + K_{02}\right)^2}} - \text{Sin}^{-1}\left(\lambda B e^{-\xi} + K_{02}\right)$$

The sixth integral is solved by using the substitution $x = \lambda B e^{-\xi}$ and the following relation (integration by parts and (Beyer, 1984:257)).

$$\int \frac{\lambda B e^{-\xi}\left(\lambda B e^{-\xi} + K_{02}\right)\xi}{\left[1 - \left(\lambda B e^{-\xi} + K_{02}\right)^2\right]^{\frac{3}{2}}} \, d\xi$$

$$= \frac{\xi}{\sqrt{1 - \left(\lambda B e^{-\xi} + K_{02}\right)^2}} - \int \frac{d\xi}{\sqrt{1 - \left(\lambda B e^{-\xi} + K_{02}\right)^2}}$$

$$\int \frac{d\xi}{\sqrt{1 - \left(\lambda B e^{-\xi} + K_{02}\right)^2}} = -\int \frac{dx}{x \sqrt{1 - \left(x + K_{02}\right)^2}}$$

$$= \frac{1}{\sqrt{1 - K_{02}^2}} \ln\left[\frac{2\sqrt{(1 - K_{02}^2)\left((1 - K_{02}^2) - 2K_{02}x - x^2\right)} - 2K_{02}x + 2(1 - K_{02}^2)}{x}\right]$$

Thus, Eq (E.4) gives the solution as $u_1 = \dfrac{\int Q(\xi)\, e^{\int P(\xi)\, d\xi} + K}{e^{\int P(\xi)\, d\xi}}$ or

141

Solution: $u_1 = \left(\lambda B e^{-\xi} + K_{02}\right)^2 \exp\left(-\frac{2}{\lambda} \text{Cos}^{-1}\left(\lambda B e^{-\xi} + K_{02}\right)\right)$

$$\times \left\{ K_{11} + \frac{2K_{01}\left(K_{12} + K_{02} K_{121}\right)(\xi + 1)}{\lambda B e^{-\xi} + K_{02}} - \frac{2K_{121} K_{01}}{\lambda} \text{Sin}^{-1}\left(\lambda B e^{-\xi} + K_{02}\right)\right.$$

$$+ \frac{\frac{2}{\lambda}\left[K_{01}K_{12} - K_{01} K_{121}\right]\left(K_{02}\xi - \lambda B e^{-\xi}\right)}{\sqrt{1 - \left(\lambda B e^{-\xi} + K_{02}\right)^2}} + \frac{2K_{02} K_{01} K_{121}}{\lambda \sqrt{1 - K_{02}{}^2}}$$

$$\left. \times \ln\left[\frac{2\sqrt{(1-K_{02}{}^2)\left[1-(\lambda B e^{-\xi} + K_{02})^2\right]} - 2K_{02}\left(\lambda B e^{-\xi} + K_{02}\right) + 2}{\lambda B e^{-\xi}}\right] \right\} \quad \text{(E.27)}$$

$\underline{d\alpha_1/d\xi \text{ Equation Solution}}$. The last three ODEs are coupled and are solved as such. First, the α_1 ODE is solved using judicious assumptions. This solution is used to derive an expression for I_1. Finally the solutions for α_1 and I_1 are used to derive an expression for Ω_1. The expression for $d\alpha_1/d\xi$ is given as Eq (E.24).

$$\frac{d\alpha_1}{d\xi} = \cot(\gamma_0) - B\delta e^{-\xi}\left[\frac{\gamma_1 \sin(\alpha_0)}{\tan(I_0)}\left(\frac{1}{\cos^2(\gamma_0)} - \frac{1}{\sin^2(\gamma_0)}\right)\right.$$

$$\left. - \frac{I_1 \sin(\alpha_0)}{\sin^2(I_0) \sin(\gamma_0) \cos(\gamma_0)} + \frac{\alpha_1\cos(\alpha_0)}{\tan(I_0) \sin(\gamma_0) \cos(\gamma_0)}\right]$$

Since re-entry vehicles enter a planetary atmosphere with small bank angles (σ), the value for δ, where by definition (Eq (3.33)) $\delta = (C_L/C_D) \sin(\sigma)$, is very small, or $\delta \sim O(0.001)$. Combining this with the fact $e^{-\xi}$ is also small, or $e^{-\xi} \sim O(0.1)$, results in a simplifying assumption for the $d\alpha_1/d\xi$ ODE. Assuming $\cot(\gamma_0) \gg B\delta e^{-\xi}$, which is valid for $\gamma_0 < 80°$ and typical

142

values for B ($B \sim O(10)$) (Hillje, 1969:2), the coupling in the $d\alpha_1/d\xi$ ODE is removed. Thus, the $d\alpha_1/d\xi$ ODE becomes

$$\frac{d\alpha_1}{d\xi} \approx \cot(\gamma_0) = \frac{\cos(\gamma_0)}{\sin(\gamma_0)}$$

This ODE is solved by first rewriting the terms $\sin(\gamma_0)$ and $\cos(\gamma_0)$ in Eq (E.8) in terms of ξ, where, $q = \cos(\gamma_0) = \lambda B e^{-\xi} + K_{02}$. Thus, $\sin(\gamma_0) = \sqrt{1-\cos^2(\gamma_0)} = \sqrt{1-(\lambda B e^{-\xi} + K_{02})^2}$. Substituting these into the above ODE gives

$$\frac{d\alpha_1}{d\xi} = \frac{\lambda B e^{-\xi} + K_{02}}{\sqrt{1-(\lambda B e^{-\xi} + K_{02})^2}}$$

Now, the Method of Separation of Variables is used to solve the above equation. Rearranging the above equation into form of Eq (E.1) gives

$$\alpha_1 = K_{15} + \int \frac{\lambda B e^{-\xi} + K_{02}}{\sqrt{1-(\lambda B e^{-\xi} + K_{02})^2}} \, d\xi$$

Using the substitution $x = \lambda B e^{-\xi}$, the above integral is reduced to

$$\alpha_1 = K_{15} - \int \frac{dx}{\sqrt{(1 - K_{02}^2) - 2K_{02}x - x^2}} - K_{02} \int \frac{dx}{x\sqrt{(1 - K_{02}^2) - 2K_{02}x - x^2}}$$

The first integral is easily solved (Beyer, 1984:255) and gives

$$\int \frac{dx}{\sqrt{(1 - K_{02}^2) - 2K_{02}x - x^2}} = \text{Sin}^{-1}\left(x + K_{02}\right)$$

With some manipulation, the second integral is easily solved (Beyer, 1984:257) and gives

143

$$\int \frac{dx}{x\sqrt{(1 - K_{02}^2) - 2K_{02}x - x^2}}$$

$$= -\frac{1}{\sqrt{1-K_{02}^2}} \ln\left[\frac{2\sqrt{(1-K_{02}^2)\left((1-K_{02}^2) - 2K_{02}x - x^2\right)} - 2K_{02}x + (1-K_{02}^2)}{x} \right]$$

Substituting the above two integrals back into the original expression for α_1 gives

Solution:
$$\alpha_1 = K_{15} - \mathrm{Sin}^{-1}\left(\lambda Be^{-\xi} + K_{02}\right) + \frac{K_{02}}{\sqrt{1-K_{02}^2}}$$

$$\times \ln\left[\frac{2\sqrt{(1-K_{02}^2)\left[1-\left(\lambda Be^{-\xi}+K_{02}\right)^2\right]} - 2K_{02}\left(\lambda Be^{-\xi}+K_{02}\right) + 2}{\lambda Be^{-\xi}} \right] \quad (E.28)$$

Figure E4 compares the exact solution of α_1, obtained from numerical integration of α_1, and the approximate function derived for α_1. While there is a minor disparity at small values of ξ (low orbital altitudes), the above difference is negligible when the magnitude of the disparity is compared to the magnitude of the functions at this altitude. This difference between the exact and approximate solutions reflect the same behavior as found in the approximate solution for q_1 (Figure E.2)

$\underline{dI_1/d\xi \text{ Equation Solution}}$. The expression for $dI_1/d\xi$ is given as Eq (E.22).

$$\frac{dI_1}{d\xi} = B\delta e^{-\xi} \left[\gamma_1\cos(\alpha_0)\left(\frac{1}{\cos^2(\gamma_0)} - \frac{1}{\sin^2(\gamma_0)}\right) - \frac{\alpha_1\sin(\alpha_0)}{\sin(\gamma_0)\cos(\gamma_0)} \right]$$
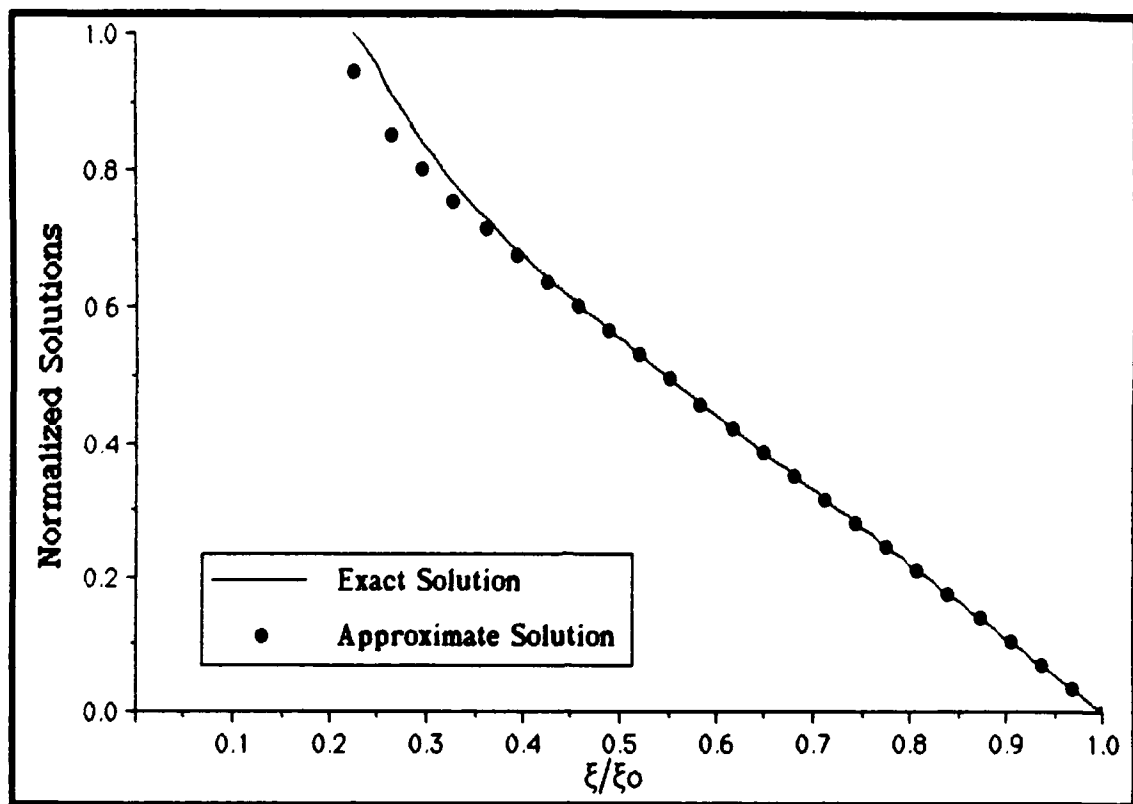
144

Figure E4.   Comparison of the Approximate and Exact Integral
Solutions for α_1

The above ODE is transformed to $dI_1/d\gamma_0$ via the chain rule for differentiation, use of Eq (E.9) and $q_0 = \cos(\gamma_0)$

$$- \lambda Be^{-\xi} = \frac{dq_0}{d\xi} = \frac{d\left(\cos(\gamma_0)\right)}{d\xi} = -\sin(\gamma_0)\frac{d\gamma_0}{d\xi} \;\Rightarrow\; \frac{d\gamma_0}{d\xi} = \frac{\lambda Be^{-\xi}}{\sin(\gamma_0)}$$

Using this relation and Eq (E.22) gives

$$\frac{dI_1}{d\gamma_0} = \frac{dI_1}{d\xi}\frac{d\xi}{d\gamma_0}$$

$$= \frac{\sin(\gamma_0)}{\lambda Be^{-\xi}}\; B\delta e^{-\xi}\left[\gamma_1\cos(\alpha_0)\left(\frac{1}{\cos^2(\gamma_0)} - \frac{1}{\sin^2(\gamma_0)}\right) - \frac{\alpha_1\sin(\alpha_0)}{\sin(\gamma_0)\cos(\gamma_0)}\right]$$

145

$$-\frac{\delta}{\lambda}\left[\gamma_1\cos(\alpha_0)\left(\frac{\sin(\gamma_0)}{\cos^2(\gamma_0)}-\frac{1}{\sin(\gamma_0)}\right)-\frac{\alpha_1\sin(\alpha_0)}{\cos(\gamma_0)}\right]$$

From Eqs (E.18), (E.25) and (E.28), relations for α_0, q_1 and α_1 are given, and by definition (Eq (D.28)), $\gamma_1 = -q_1/\sin(\gamma_0)$. Substituting these relations into the above ODE gives

$$\frac{dI_1}{d\gamma_0}=\frac{\delta}{\lambda}\left[\gamma_1\cos(\alpha_0)\left(\frac{1}{\sin^2(\gamma_0)}-\frac{1}{\cos^2(\gamma_0)}\right)-\frac{\alpha_1\sin(\alpha_0)}{\cos(\gamma_0)}\right]$$

$$=\frac{\delta}{\lambda}\left\{\left[K_{121}\left(K_{02}-K_{02}\ln\left(\frac{\cos(\gamma_0)-K_{02}}{\lambda B}\right)-\cos(\gamma_0)\right)+K_{12}\right]\right.$$

$$\times\left(\frac{1}{\sin^2(\gamma_0)}-\frac{1}{\cos^2(\gamma_0)}\right)\frac{\cos(K_{03})\sin\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4}+\frac{\gamma_0}{2}\right)\right)+K_{05}\right]}{\sqrt{1-\left[\cos(K_{03})\cos\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4}+\frac{\gamma_0}{2}\right)\right)+K_{05}\right]\right]^2}}$$

$$-\frac{\sin(K_{03})}{\cos(\gamma_0)\sqrt{1-\left[\cos(K_{03})\cos\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4}+\frac{\gamma_0}{2}\right)\right)+K_{05}\right]\right]^2}}$$

$$\left.\times\left[K_{15}+\gamma_0-\frac{\pi}{2}+\frac{K_{02}}{\sqrt{1-K_{02}^2}}\ln\left[\frac{2\sqrt{1-K_{02}^2}\sin(\gamma_0)-2K_{02}\cos(\gamma_0)+2}{\cos(\gamma_0)-K_{02}}\right]\right]\right\}$$

Before separation of variables is used to solve the above equation, several approximations are used to bring the integration task down to a manageable level. First, using order of magnitude analysis involving δ/λ,

146

and K_{05}, $\delta/\lambda \sim O(0.001)$ and $K_{05} \gg \delta/\lambda$. Therefore, the following approximation is made.

$$\sin\left[\frac{\delta}{\lambda} \ln\left(\tan\left(\frac{\pi}{4} + \frac{\gamma_0}{2}\right)\right) + K_{05}\right] \sim \sin(K_{05})$$

Therefore

$$\cos\left[\frac{\delta}{\lambda} \ln\left(\tan\left(\frac{\pi}{4} + \frac{\gamma_0}{2}\right)\right) + K_{05}\right] \sim \cos(K_{05})$$

Also, through numerical analysis, the following approximations are made.

$$\ln\left(\cos(\gamma_0) - K_{02}\right) \sim -\cos(\gamma_0)$$

$$\frac{K_{02}}{\sqrt{1 - K_{02}{}^2}} \ln\left(2\sqrt{1 - K_{02}{}^2}\, \sin(\gamma_0) - 2\, K_{02} \cos(\gamma_0) + 2\right) \sim 0$$

Substituting these relations into the above ODE and combining integrands gives

$$\frac{dI_1}{d\gamma_0} \sim \frac{\delta}{\lambda}\left\{\left[K_{121}\left(K_{02}\left(1 + \ln(\lambda B)\right) - (1 + K_{02})\cos(\gamma_0)\right) + K_{12}\right]\right.$$

$$\times \left(\frac{1}{\sin^2(\gamma_0)} - \frac{1}{\cos^2(\gamma_0)}\right) \frac{\sin(K_{05})\cos(K_{03})}{\sqrt{1 - \sin^2(K_{05})\cos^2(K_{03})}}$$

$$- \frac{\sin(K_{03})}{\cos(\gamma_0)\sqrt{1 - \sin^2(K_{05})\cos^2(K_{03})}}$$

$$\times \left.\left[\frac{K_{02}}{\sqrt{1-K_{02}{}^2}}\cos(\gamma_0) + K_{15} + \gamma_0 - \frac{\pi}{2}\right]\right\}$$

147

Rearranging the above equation into form of Eq (E.1) gives

$$
I_1 - K_{13} + \frac{\delta}{\lambda} \left\{ \frac{K_{02} \sin(K_{03})}{\sqrt{1 - K_{02}^2} \sqrt{1 - \sin^2(K_{05}) \cos^2(K_{03})}} \int d\gamma_0 \right.
$$

$$
- \frac{\sin(K_{05}) K_{121} \left(K_{02} + 1 \right) \cos(K_{03})}{\sqrt{1 - \sin^2(K_{05}) \cos^2(K_{03})}} \int \frac{\cos(\gamma_0)}{\sin^2(\gamma_0)} d\gamma_0
$$

$$
+ \frac{\sin(K_{05}) \left[K_{12} + K_{02} K_{121} \left(1 + \ln(\lambda B) \right) \right] \cos(K_{03})}{\sqrt{1 - \sin^2(K_{05}) \cos^2(K_{03})}} \int \frac{d\gamma_0}{\sin^2(\gamma_0)}
$$

$$
- \frac{\sin(K_{05}) \left[K_{12} + K_{02} K_{121} \left(1 + \ln(\lambda B) \right) \right] \cos(K_{03})}{\sqrt{1 - \sin^2(K_{05}) \cos^2(K_{03})}} \int \frac{d\gamma_0}{\cos^2(\gamma_0)}
$$

$$
+ \left[\frac{\sin(K_{05}) K_{121} \cos(K_{03}) \left(K_{02} + 1 \right) + \sin(K_{03}) \left(\frac{\pi}{2} - K_{15} \right)}{\sqrt{1 - \sin^2(K_{05}) \cos^2(K_{03})}} \right] \int \frac{d\gamma_0}{\cos(\gamma_0)}
$$

$$
\left. - \frac{\sin(K_{03})}{\sqrt{1 - \sin^2(K_{05}) \cos^2(K_{03})}} \int \frac{\gamma_0 \, d\gamma_0}{\cos(\gamma_0)} \right\}
$$

The first five integrals are easily solved (Beyer, 1984:260,261). The sixth integral is given by (Gradshteyn and Ryzhik, 1980:189)

$$
\int \frac{x^n \, dx}{\cos(x)} - \sum_{k=0}^{\infty} \frac{|E_{2k}| \, x^{2k+n+1}}{(2k+n+1) \, (2k!)}
$$

148

The term $|E_{2k}|$ is the Euler constant: its first few terms are given as (Gradshteyn and Ryzhik, 1980:xxix) $|E_0| = 1$, $|E_2| = 1$ and $|E_4| = 5$. Thus the above integral becomes

$$\int \frac{y_0 \, dy_0}{\cos(y_0)} = \frac{y_0^2}{2} + \frac{y_0^4}{8} + \frac{5 y_0^6}{144} + O(y_0^8) = \frac{y_0^2}{2} + \frac{y_0^4}{8}$$

Substituting $y_0 = \text{Cos}^{-1}\left(\lambda B e^{-\xi} + K_{02}\right)$ gives the solution for I_1 as

$$I_1 = K_{13} + \frac{\delta}{\lambda}\left\{ \frac{K_{131}\left(K_{132} - \dfrac{K_{133}}{\cos(y_0)}\right)}{\sin(y_0)} + K_{134}\left[\frac{K_{02}\, y_0}{\sqrt{1 - K_{02}^2}} - \frac{y_0^2}{2} - \frac{y_0^4}{8} \right] \right.$$

$$\left. + \left[K_{131}K_{132} + K_{134}\left(\frac{\pi}{2} - K_{15}\right) \right] \ln\left[\tan\left(\frac{\pi}{4} + \frac{y_0}{2}\right) \right] \right\}$$

Solution: $I_1 = K_{13} + \dfrac{\delta}{\lambda}\left\{ \dfrac{K_{131}\left(K_{132} - \dfrac{K_{133}}{\lambda B e^{-\xi} + K_{02}}\right)}{\sqrt{1 - \left(\lambda B e^{-\xi} + K_{02}\right)^2}} \right.$

$$+ \left[K_{131}K_{132} + K_{134}\left(\frac{\pi}{2} - K_{15}\right) \right] \ln\left[\tan\left(\frac{\pi}{4} + \frac{\text{Cos}^{-1}\left(\lambda B e^{-\xi} + K_{02}\right)}{2}\right) \right]$$

$$+ K_{134}\left[\frac{K_{02}\, \text{Cos}^{-1}\left(\lambda B e^{-\xi} + K_{02}\right)}{\sqrt{1 - K_{02}^2}} - \frac{\left[\text{Cos}^{-1}\left(\lambda B e^{-\xi} + K_{02}\right)\right]^2}{2} \right]$$

149

$$-\frac{\left[Cos^{-1}\left(\lambda Be^{-\xi}+K_{02}\right)\right]^4}{8}\right]\right] \tag{E.29}$$

where $K_{131} = \dfrac{\sin(K_{05})\cos(K_{03})}{\sqrt{1-\sin^2(K_{05})\cos^2(K_{03})}}$

$K_{132} = K_{121}\left(1+K_{02}\right)$

$K_{133} = K_{12} + K_{02}K_{121}\left(1+\ln(\lambda B)\right)$

$$K_{134} = \frac{\sin(K_{03})}{\sqrt{1-\sin^2(K_{05})\cos^2(K_{03})}} \tag{E.30}$$

Figure E5 compares the relationship between the exact solution of I_1, derived by numerical integration, and the approximate, analytical solution derived above. As in Figure E3, the correlation between exact and approximate solutions is good, with a small difference appearing as $\xi \to 0$. This is the same phenomena found in the approximation for q_1. Thus, even with the three approximations used to derive this solution, a significant portion of the disparity between the exact and approximate solutions are attributable to the approximate expression used for q_1.

$d\Omega_1/d\xi$ Equation Solution. The expression for $d\Omega_1/d\xi$ is given as Eq (E.23).

$$\frac{d\Omega_1}{d\xi} = B\delta e^{-\xi}\left[\frac{y_1\sin(\alpha_0)}{\sin(I_0)}\left(\frac{1}{\cos^2(y_0)}-\frac{1}{\sin^2(y_0)}\right)\right.$$

$$\left.-\frac{I_1\sin(\alpha_0)\cos(I_0)}{\sin^2(I_0)\sin(y_0)\cos(y_0)}+\frac{\alpha_1\cos(\alpha_0)}{\sin(I_0)\sin(y_0)\cos(y_0)}\right]$$
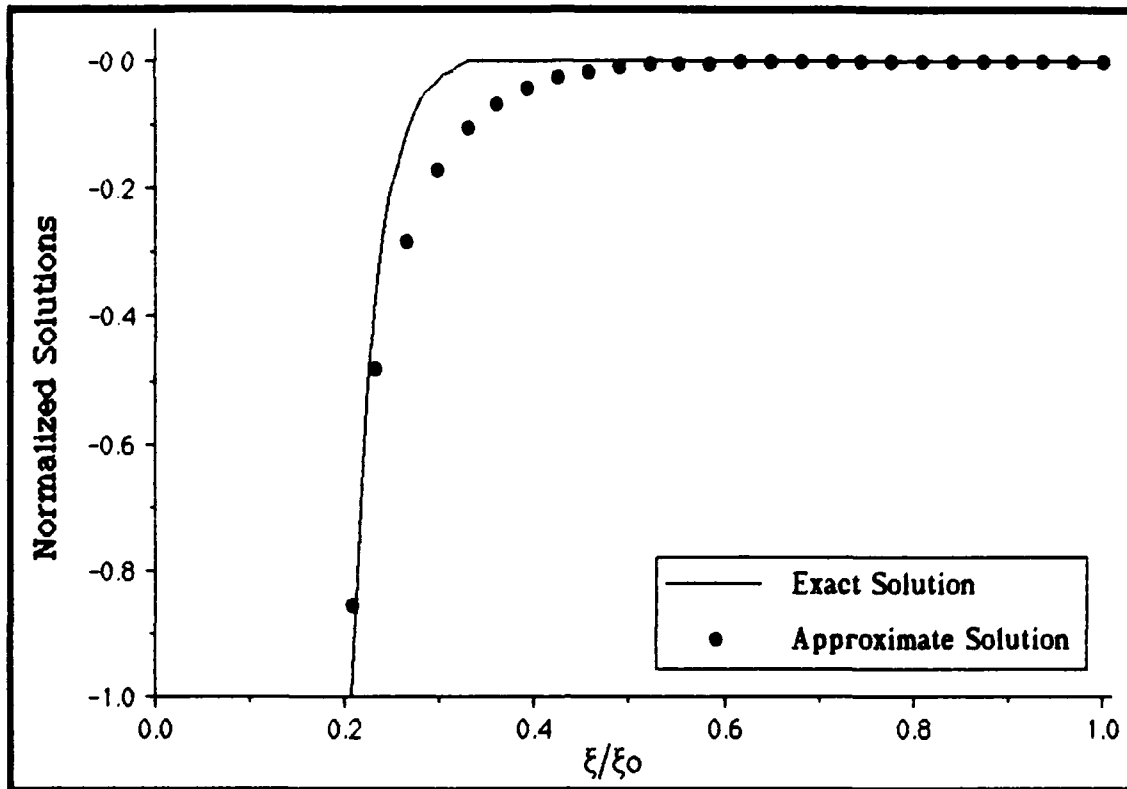
150

Figure E5.   Comparison of the Approximate and Exact Integral
Solutions for I_1

The above ODE is transformed to $d\Omega_1/dy_0$ via the chain rule for

differentiation, use of Eq (E.9) and $q_0 = \cos(y_0)$

$$-\lambda Be^{-\xi} = \frac{dq_0}{d\xi} = \frac{d\left(\cos(y_0)\right)}{d\xi} = -\sin(y_0)\frac{dy_0}{d\xi} \rightarrow \frac{dy_0}{d\xi} = \frac{\lambda Be^{-\xi}}{\sin(y_0)}$$

Using this relation and Eq (E.23) gives

$$\frac{d\Omega_1}{dy_0} = \frac{d\Omega_1}{d\xi}\frac{d\xi}{dy_0}$$

151

$$= \frac{\delta}{\lambda} \left[\frac{\gamma_1 \sin(\gamma_0) \sin(\alpha_0)}{\sin(I_0)} \left(\frac{1}{\cos^2(\gamma_0)} - \frac{1}{\sin^2(\gamma_0)} \right) \right.$$

$$\left. + \frac{1}{\sin(I_0) \cos(\gamma_0)} \left(\alpha_1 \cos(\alpha_0) - \frac{I_1 \sin(\alpha_0) \cos(I_0)}{\sin(I_0)} \right) \right]$$

From Eqs (E.18), (E.25), (E.28) and (E.29), relations for α_0, q_1, α_1 and I_1 are given, and by definition (Eq (D.28)), $\gamma_1 = -q_1/\sin(\gamma_0)$. Substituting these relations into the above ODE gives

$$\frac{d\Omega_1}{d\gamma_0} = \frac{d\Omega_1}{d\xi} \frac{d\xi}{d\gamma_0}$$

$$= \frac{\delta}{\lambda} \left[\frac{\gamma_1 \sin(\gamma_0) \sin(\alpha_0)}{\sin(I_0)} \left(\frac{1}{\cos^2(\gamma_0)} - \frac{1}{\sin^2(\gamma_0)} \right) \right.$$

$$\left. + \frac{1}{\sin(I_0) \cos(\gamma_0)} \left(\alpha_1 \cos(\alpha_0) - \frac{I_1 \sin(\alpha_0) \cos(I_0)}{\sin(I_0)} \right) \right]$$

$$= \frac{\delta}{\lambda} \left\{ \frac{\left[K_{133} - K_{02} K_{121} \ln\left(\cos(\gamma_0) - K_{02} \right) - K_{121} \cos(\gamma_0) \right] \left(\frac{\sin(K_{03})}{\sin^2(\gamma_0)} - \frac{\sin(K_{03})}{\cos^2(\gamma_0)} \right)}{1 - \left[\cos(K_{03}) \cos\left[\frac{\delta}{\lambda} \ln\left(\tan\left(\frac{\pi}{4} + \frac{\gamma_0}{2} \right) \right) + K_{05} \right] \right]^2} \right.$$

$$\left. + \frac{K_{15} + \gamma_0 - \frac{\pi}{2} + \frac{K_{02}}{\sqrt{1 - K_{02}^2}} \ln\left[\frac{2\sqrt{1 - K_{02}^2} \sin(\gamma_0) - 2K_{02} \cos(\gamma_0) + 2}{\cos(\gamma_0) - K_{02}} \right]}{\cos(\gamma_0) \sqrt{1 - \left[\cos(K_{03}) \cos\left[\frac{\delta}{\lambda} \ln\left(\tan\left(\frac{\pi}{4} + \frac{\gamma_0}{2} \right) \right) + K_{05} \right] \right]^2}} \right.$$

$$\times \left(1 - \frac{\sin^2\left(K_{03}\right)}{1 - \left[\cos(K_{03})\cos\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4}+\frac{\gamma_0}{2}\right)\right)+K_{05}\right]\right]^2}\right)^{\frac{1}{2}}$$

$$+ \frac{\sin(K_{03})\cos(K_{03})\cos\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4}+\frac{\cos^{-1}\left(\lambda Be^{-\xi}+K_{02}\right)}{2}\right)\right)+K_{05}\right]}{\cos(\gamma_0)\left[1 - \left[\cos(K_{03})\cos\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4}+\frac{\gamma_0}{2}\right)\right)+K_{05}\right]\right]^2\right]^{\frac{3}{2}}}$$

$$\times \left(K_{13} + \frac{\delta}{\lambda}\left(\frac{K_{131}\left(K_{132}-\frac{K_{133}}{\cos(\gamma_0)}\right)}{\sin(\gamma_0)} + K_{134}\left[\frac{K_{02}\,\gamma_0}{\sqrt{1-K_{02}^2}} - \frac{\gamma_0^2}{2} - \frac{\gamma_0^4}{8}\right]\right.\right.$$

$$\left.\left. + \left[K_{131}K_{132} + K_{134}\left(\frac{\pi}{2}-K_{15}\right)\right]\ln\left[\tan\left(\frac{\pi}{4}+\frac{\gamma_0}{2}\right)\right]\right)\right)$$

Before separation of variables is used to solve the above equation, several approximations are used to bring the integration task down to a manageable level. First, using numerical analysis involving δ/λ and K_{05}, $\delta/\lambda \sim O(0.001)$ and $K_{05} \gg \delta/\lambda$. Therefore, the following approximation is made.

$$\sin\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4}+\frac{\gamma_0}{2}\right)\right)+K_{05}\right] \sim \sin(K_{05})$$

Therefore

$$\cos\left[\frac{\delta}{\lambda}\ln\left(\tan\left(\frac{\pi}{4}+\frac{\gamma_0}{2}\right)\right)+K_{05}\right] \sim \cos(K_{05})$$

153

Also, through numerical analysis, the following approximations are made.

$$\ln\left(\cos(\gamma_0) - K_{02}\right) \approx -\cos(\gamma_0)$$

$$\frac{K_{02}}{\sqrt{1 - K_{02}^2}} \ln\left(2\sqrt{1 - K_{02}^2}\,\sin(\gamma_0) - 2\,K_{02}\cos(\gamma_0) + 2\right) \approx 0$$

Substituting these relations into the above ODE ,combining integrands and rearranging the result into the form of Eq (E.1) gives (where A_1 - A_{11} are constants)

$$\Omega_1 = K_{14} + \frac{\delta}{\lambda}\left[A_1 \int \frac{d\gamma_0}{\sin^2(\gamma_0)} + A_2 \int \frac{d\gamma_0}{\cos^2(\gamma_0)} + A_3 \int \frac{\cos(\gamma_0)\,d\gamma_0}{\sin(\gamma_0)}\right.$$

$$+ A_4 \int \frac{d\gamma_0}{\cos(\gamma_0)} + A_5 \int \frac{d\gamma_0}{\cos(\gamma_0)\sin(\gamma_0)} + A_6 \int \frac{d\gamma_0}{\sin(\gamma_0)\cos^2(\gamma_0)}$$

$$+ A_7 \int \frac{\gamma_0\,d\gamma_0}{\cos(\gamma_0)} + A_8 \int \frac{\gamma_0^2\,d\gamma_0}{\cos(\gamma_0)} + A_9 \int \frac{\gamma_0^4\,d\gamma_0}{\cos(\gamma_0)} + A_{10}\int d\gamma_0$$

$$\left.+ A_{11} \int \frac{\ln\left[\tan\left(\frac{\pi}{4} + \frac{\gamma_0}{2}\right)\right]d\gamma_0}{\cos(\gamma_0)}\right.$$

The first six integrals are easily solved (Beyer, 1984:260,261,263). The seventh through the ninth integrals are given by (Gradshteyn and Ryzhik, 1980:189)

$$\int \frac{x^n\,dx}{\cos(x)} = \sum_{k=0}^{\infty} \frac{|E_{2k}|\,x^{2k+n+1}}{(2k+n+1)\,(2k!)}$$

154

The term $|E_{2k}|$ is the Euler constant: its first few terms are given as (Gradshteyn and Ryzhik, 1980:xxix) $|E_0| = 1$, $|E_2| = 1$ and $|E_4| = 5$. Thus the above integrals becomes

$$\int \frac{y_0^n \, dy_0}{\cos(y_0)} = \frac{y_0^{n+1}}{n+1} + \frac{y_0^{n+3}}{2(n+3)} + \frac{5y_0^{n+5}}{24(n+5)} + O(y_0^{n+7})$$

The eleventh integral is transformed to put the equation in a suitable form. By using trigonometric identities (Beyer, 1984:138)

$$\tan\left(\frac{\pi}{4} + \frac{y_0}{2}\right) = \frac{\cos(y_0)}{1 - \sin(y_0)}$$

Using the law of logarithms

$$\ln\left[\tan\left(\frac{\pi}{4} + \frac{y_0}{2}\right)\right] = \ln\left[\frac{\cos(y_0)}{1 - \sin(y_0)}\right] = \frac{1}{2}\ln\left[\frac{1 + \sin(y_0)}{1 - \sin(y_0)}\right]$$

Using the series expansion for the right hand side term yields (Beyer, 1984:297)

$$\frac{1}{2}\ln\left[\frac{1 + \sin(y_0)}{1 - \sin(y_0)}\right] = \sum_{n=1}^{\infty} \frac{\sin^{2n-1}(y_0)}{2n-1} = \sin(y_0) + \frac{\sin^3(y_0)}{3} + O(\sin^5(y_0))$$

Therefore

$$\int \frac{\ln\left[\tan\left(\frac{\pi}{4} + \frac{y_0}{2}\right)\right] dy_0}{\cos(y_0)} = \int \left(\tan(y_0) + \frac{\sin^3(y_0)}{3\cos(y_0)}\right) dy_0$$

$$= \frac{4\ln\left(\cos(y_0)\right)}{3} - \frac{\cos^2(y_0)}{6}$$

Solving the above integrals and substituting $\gamma_0 = \text{Cos}^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)$ gives the solution for I_1 as

$$\Omega_1 = K_{14} + \frac{\delta}{\lambda}\left\{ \frac{K_{141}\left(K_{132} - \dfrac{K_{133}}{\cos(\gamma_0)}\right)}{\sin(\gamma_0)} + \frac{K_{142}\,K_{133}}{\cos(\gamma_0)} + K_{143}\,\ln\left[\tan\left(\frac{\pi}{4} + \frac{\gamma_0}{2}\right)\right] \right.$$

$$+ \left(K_{132}\,K_{142} + K_{144}\left(\frac{\pi}{2} - K_{15}\right)\right)\left[\frac{4\,\ln(\cos(\gamma_0))}{3} - \frac{\cos^2(\gamma_0)}{6}\right]$$

$$\left. + \frac{K_{02}\,K_{144}\,\gamma_0}{\sqrt{1 - K_{02}{}^2}} + K_{145}\left[\frac{\gamma_0{}^3}{6} + \frac{3\gamma_0{}^5}{40}\right] + K_{146}\left[\frac{\gamma_0{}^2}{2} + \frac{\gamma_0{}^4}{8}\right]\right\}$$

Solution:

$$\Omega_1 = K_{14} + \frac{\delta}{\lambda}\left\{ \frac{K_{141}\left(K_{132} - \dfrac{K_{133}}{\lambda Be^{-\xi} + K_{02}}\right)}{\sqrt{1 - \left(\lambda Be^{-\xi} + K_{02}\right)^2}} + \frac{K_{142}\,K_{133}}{\lambda Be^{-\xi} + K_{02}} \right.$$

$$+ K_{143}\,\ln\left[\tan\left(\frac{\pi}{4} + \frac{\text{Cos}^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)}{2}\right)\right]$$

$$+ \left(K_{132}\,K_{142} + K_{144}\left(\frac{\pi}{2} - K_{15}\right)\right)\left[\frac{4\,\ln\left(\lambda Be^{-\xi} + K_{02}\right)}{3} - \frac{\left(\lambda Be^{-\xi} + K_{02}\right)^2}{6}\right]$$

$$+ \frac{K_{02}\,K_{144}\,\text{Cos}^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)}{\sqrt{1 - K_{02}{}^2}}$$

$$+ K_{145}\left[\frac{\left[Cos^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)\right]^3}{6} + \frac{3\left[Cos^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)\right]^5}{40}\right]$$

$$+ K_{146}\left[\frac{\left[Cos^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)\right]^2}{2} + \frac{\left[Cos^{-1}\left(\lambda Be^{-\xi} + K_{02}\right)\right]^4}{8}\right]\right\} \tag{E.31}$$

where $K_{141} = \dfrac{K_{134}^2}{\sin(K_{03})}$

$$K_{142} = \frac{\delta \sin(K_{05}) \cos(K_{05}) \sin(K_{03}) \cos^2(K_{03})}{\lambda \left(1 - \sin^2(K_{05}) \cos^2(K_{03})\right)^2}$$

$$K_{143} = K_{142}\left(K_{133} - K_{132}\right)$$

$$+ \frac{K_{134}\cos(K_{03})\left(\sin(K_{05}) - K_{13}\sqrt{1 - K_{02}^2}\right)}{1 - \sin^2(K_{05}) \cos^2(K_{03})}$$

$$+ \frac{\sin(K_{05}) \cos(K_{03})\left(K_{15} - \frac{\pi}{2}\right)}{1 - \sin^2(K_{05}) \cos^2(K_{03})}$$

$$K_{144} = \frac{\sin(K_{05}) \cos(K_{03})}{1 - \sin^2(K_{05}) \cos^2(K_{03})}$$

$$K_{145} = \frac{K_{142} \tan(K_{03})}{\sin(K_{05})}$$

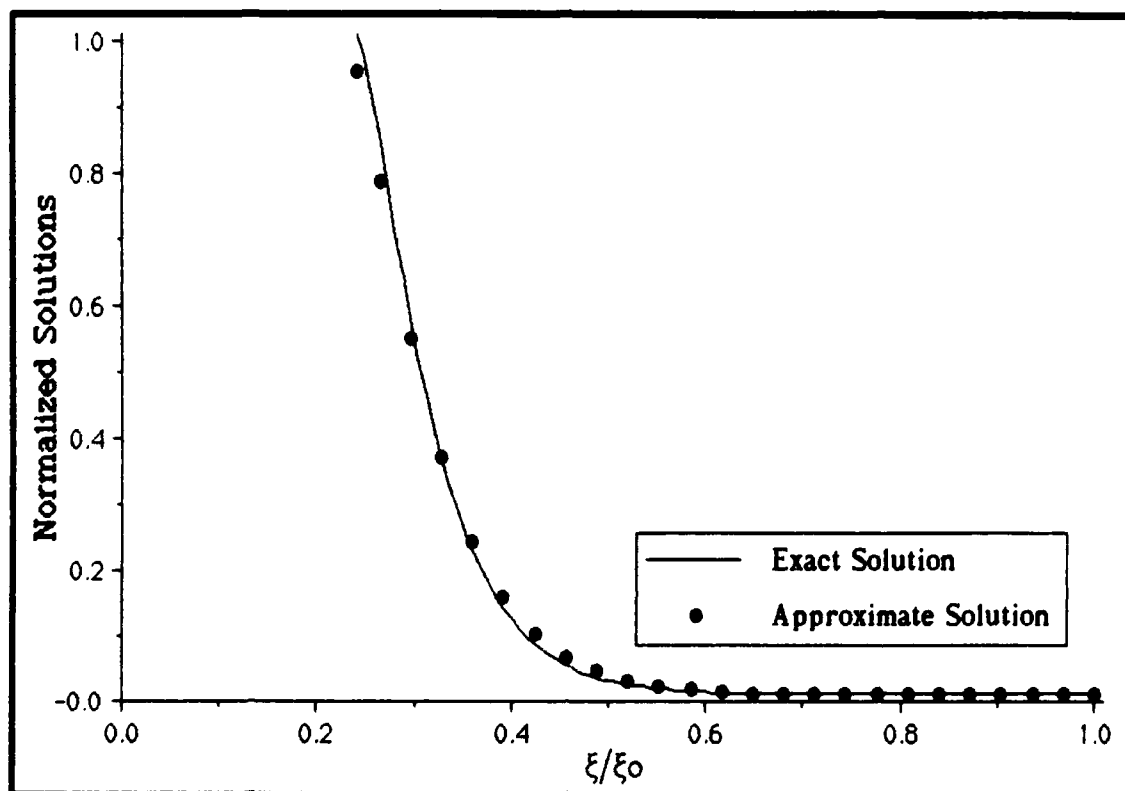$$K_{146} = K_{144} - \frac{K_{02}K_{145}}{\sqrt{1 - K_{02}^2}} \tag{E.32}$$

Figure E6.   Comparison of the Approximate and Exact Integral
Solutions for Ω_1

Figure E6 compares the relationship between the exact solution of Ω_1, derived by numerical integration, and the approximate, analytical solution derived above. As in Figure E3, the correlation between exact and approximate solutions is good, with a small difference appearing as $\xi \to 0$. This is the same phenomena found in the approximation for q_1. Thus, even with the three approximations used to derive this solution, a significant portion of the disparity between the exact and approximate solutions are attributable to the approximate expression used for q_1.

Appendix F:
Sample Earth Atmospheric Entry Trajectories

This appendix presents an example Earth atmospheric trajectory to demonstrate the use of the zero and first order composite expansions derived in Section IV, Eqs (4.107)-(4.111). The initial conditions are for an Apollo-type reentry vehicle and are derived from theoretical and flight test data (Hillje, 1969:2-10)

For the following sample trajectory presented in Figures F1-F5, the independent and dependent variables are defined in Sections II and III and are repeated below.

> y - Vehicle Altitude (m)
>
> u - Non-dimensional Speed Ratio, $u \equiv V^2 \cos^2(\gamma)/g_s r_s$
>
> γ - Flight Path Angle (deg)
>
> I - Inclination Angle (deg)
>
> Ω - Longitude of the Ascending Node (deg)
>
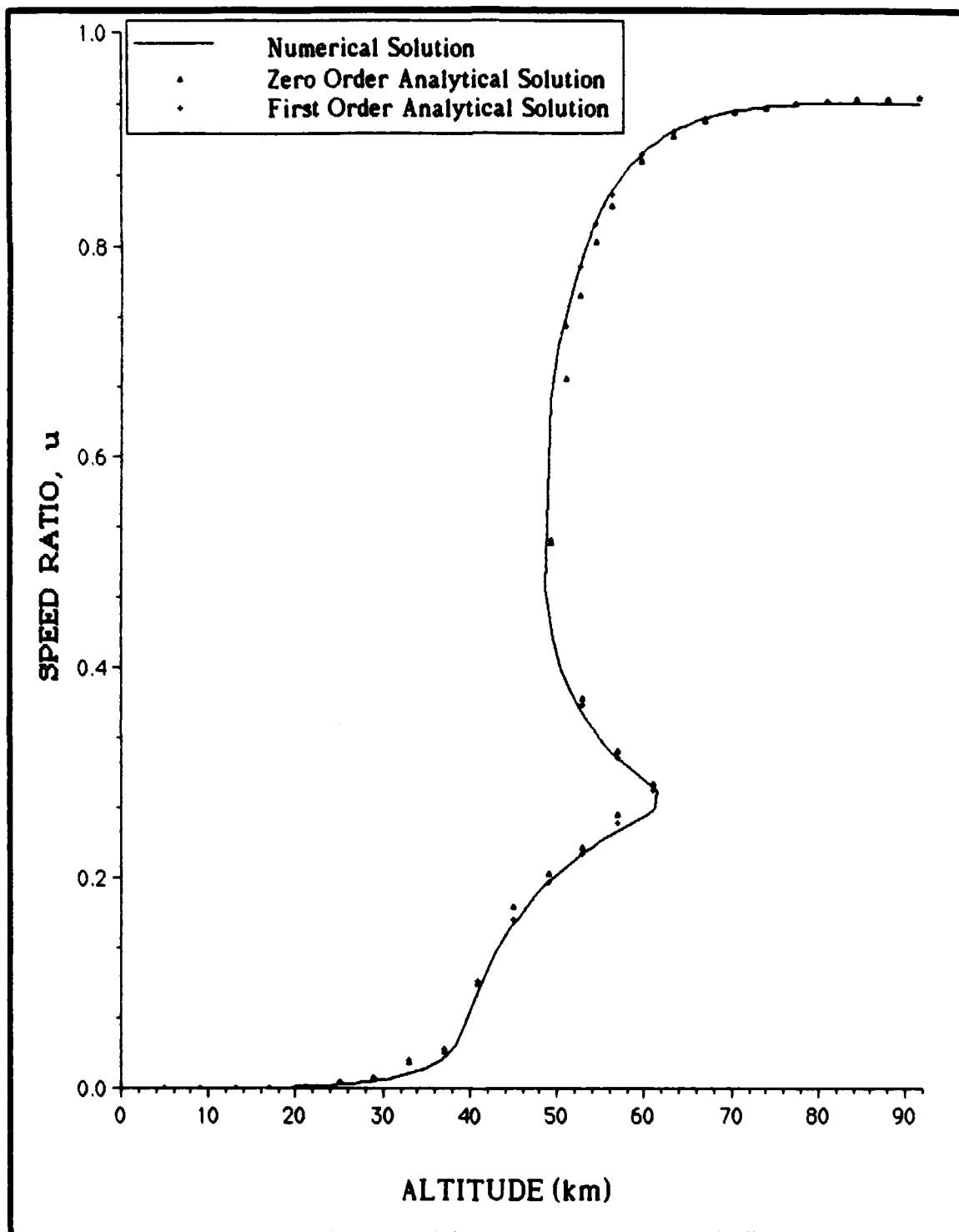> α - Latitude at Epoch (deg)

Figure F1. Comparison of the Numerical and Analytical
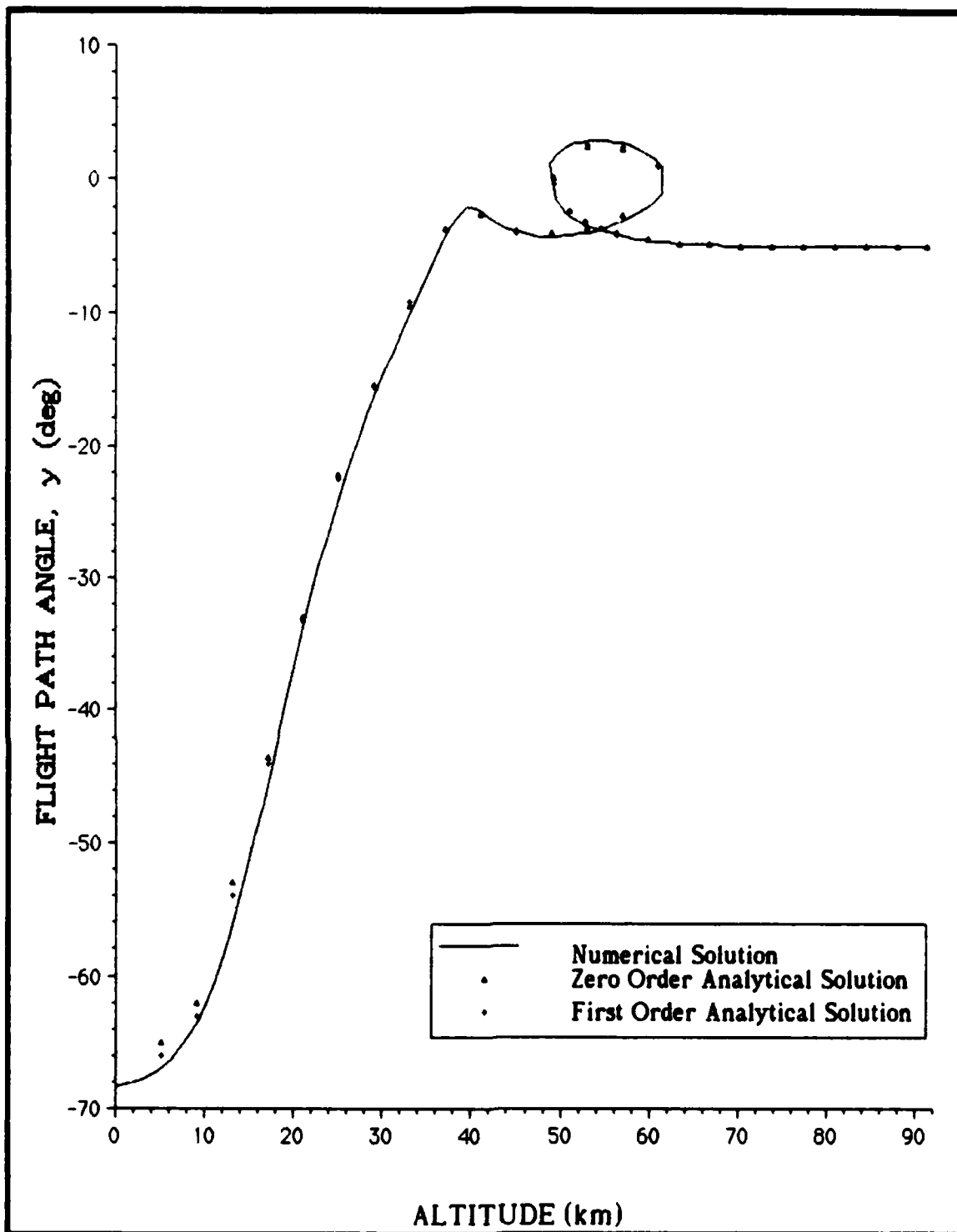Solutions for the Speed Ratio, u

160

Figure F2. Comparison of the Numerical and Analytical
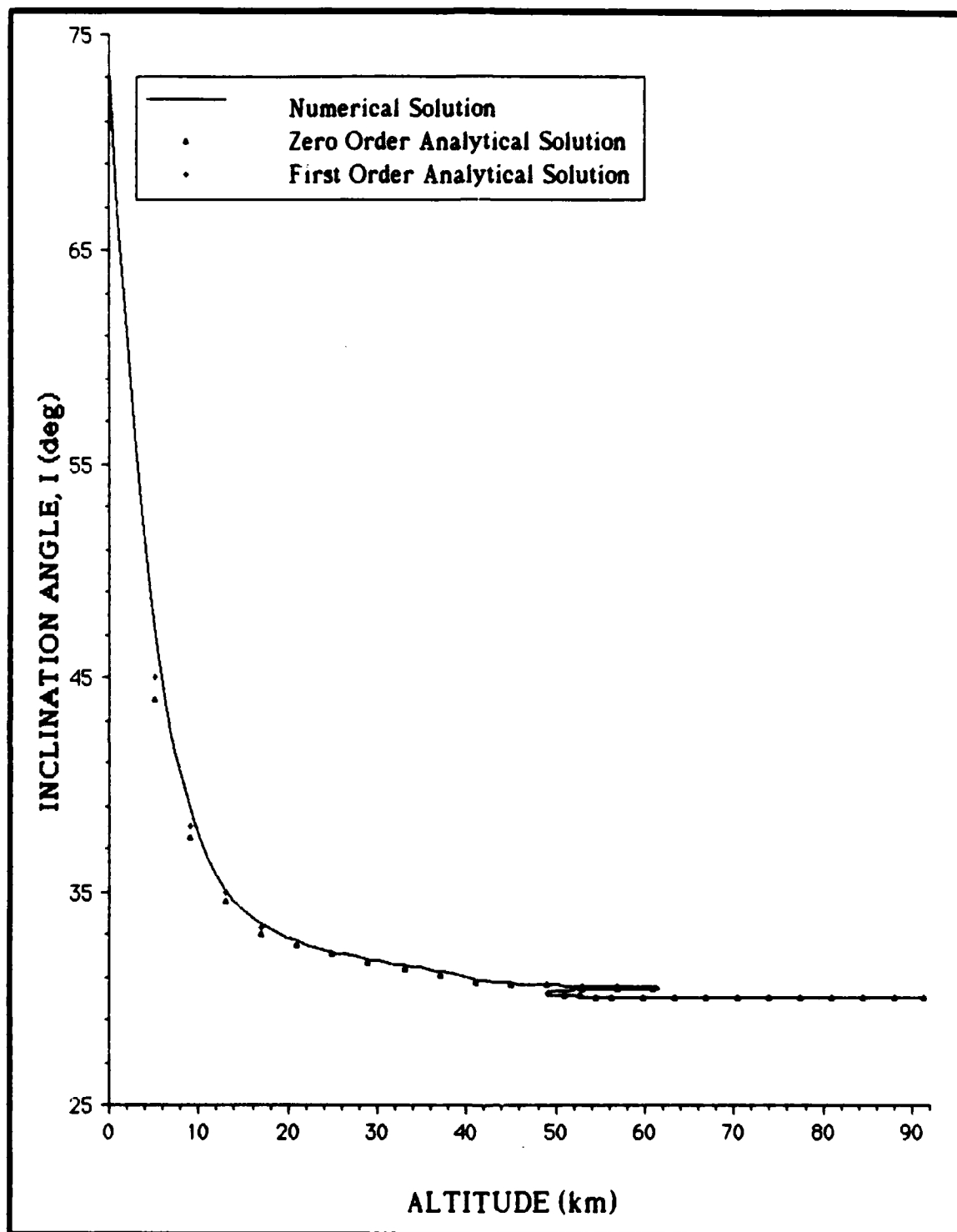Solutions for the Modified Flight Path Angle, q

Figure F3. Comparison of the Numerical and Analytical
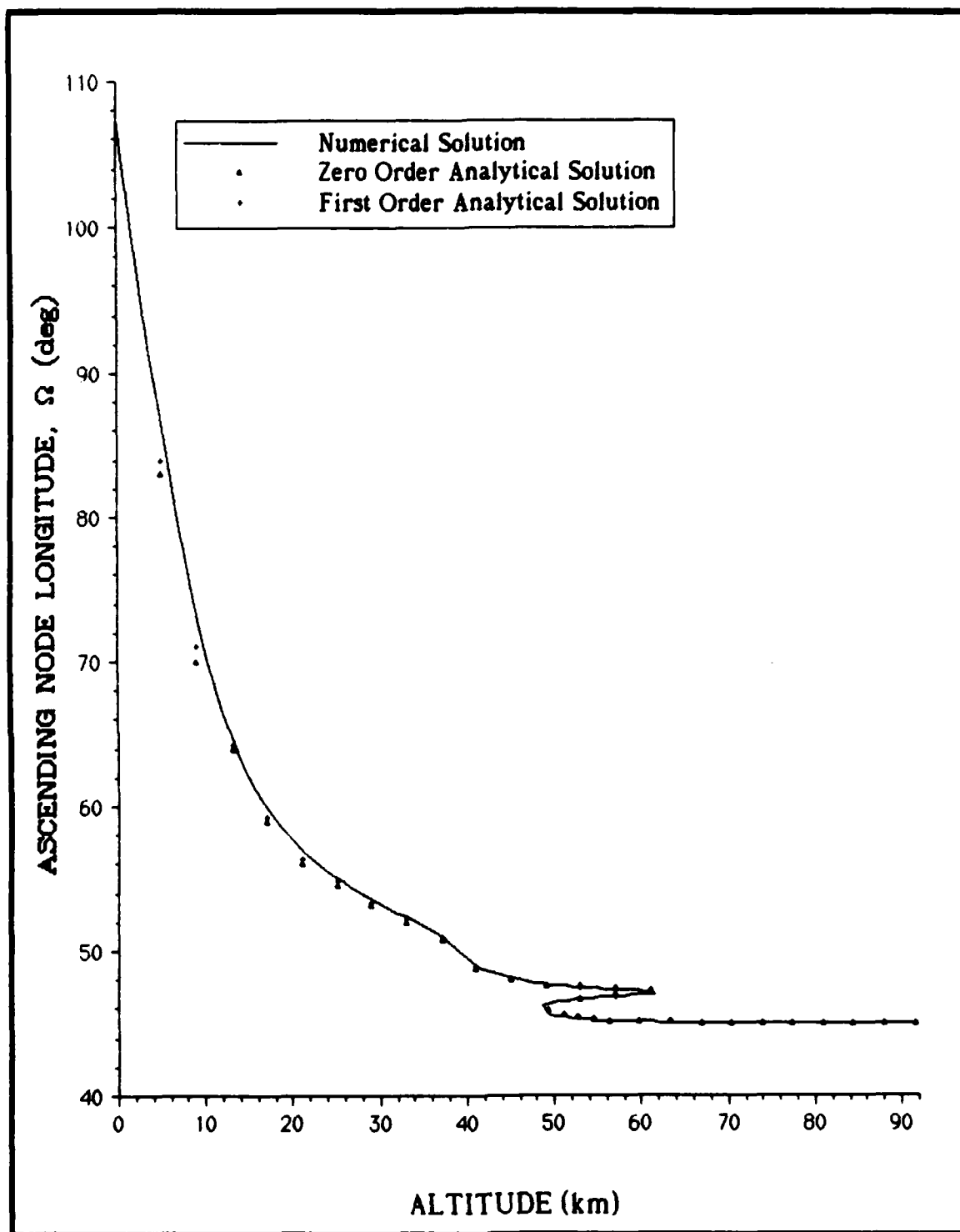Solutions for the Inclination Angle, I

Figure F4. Comparison of the Numerical and Analytical
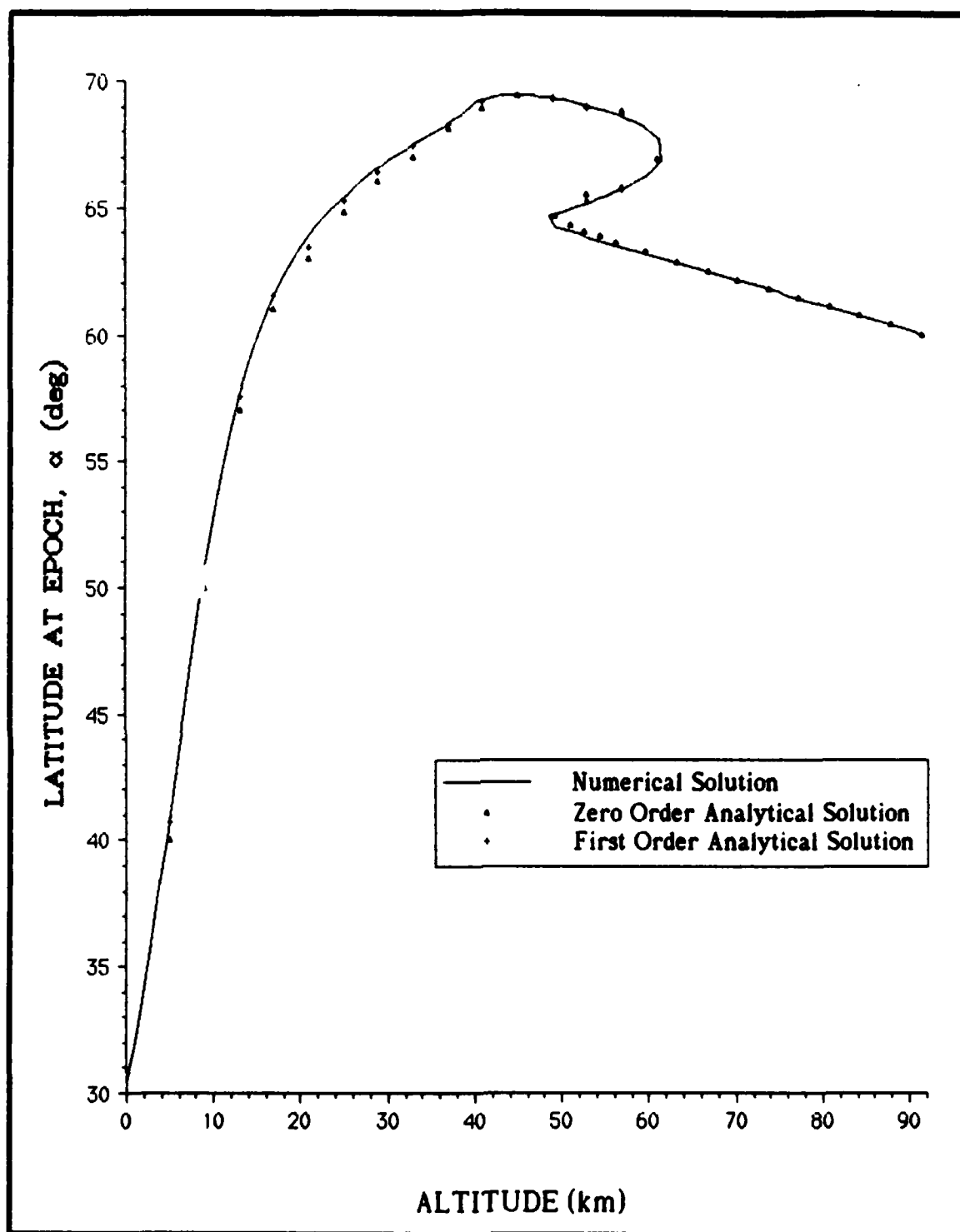Solutions for the Longitude of the Ascending Node, Ω

163

Figure F5. Comparison of the Numerical and Analytical
Solutions for the Latitude of Epoch, α

164

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Vita

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He attended Syracuse University (Syracuse, NY), where he received the degree of Bachelor of Science in Aerospace Engineering in May 1985. Upon graduation, he received a commission in the USAF through AFROTC and was assigned to Wright-Patterson AFB, OH. During his first assignment, he served as a depot activation manager for the Ground Launched Cruise Missile (GLCM) program and a logistics support manager for a F-111 flight control system upgrade program. He began part-time graduate work in the School of Engineering, Air Force Institute of Technology, in September 1985 and completed his work as a full-time student, starting in May 1988.

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Although numerical techniques exist to obtain solutions to highly non-linear and highly coupled systems, the trends and subtleties of the solution are frequently lost in the volume and form of tabular and graphical data in covering a wide range of initial conditions. By deriving an approximate, analytical solution, relationships between dependent parameters are discernable. Also, the derived solution is easily applied to any new set of initial conditions or can be modified to incorporate a slightly different system. This study presents an analytical investigation of the three-dimensional equations of motion for lifting entry into a planetary atmosphere.

In this study, the equations of motion for lifting entry into a planetary atmosphere are derived. A non-rotating, spherical planet is assumed, as is a non-rotating, strictly exponential atmospheric model. The derived equations of motion are transformed to a variable set relating the classical orbital elements to the vehicle's altitude. Solutions to the resulting five non-linear, coupled, first order, ordinary differential equations are obtained by using the Method of Matched Asymptotic Expansions and a computerized symbolic manipulator, which performs the detailed algebraic computations. By using the planetary scale height-mean equatorial radius (PSHMER) product as a small parameter, both zero and first order expansions to the equations of motion are obtained.

It is demonstrated the analytical solution agrees with results obtained from numerical integration of the equations of motion. Due to approximations made in the solutions of the first order inner expansions, the analytical solution slightly deviates from the numerical solution at low vehicle altitudes. The two solutions are compared further and the validity of the analytical solution is examined.